Advances in Applied Mathematics 46 (2011) 175–191

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Excedances of affine permutations

Eric Clark, Richard Ehrenborg *

Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027, United States

ARTICLE INFO

Article history:
Available online 15 October 2010

MSC:
05A05
05A15
20F55

Keywords:
Affine Weyl group of type *A*
Juggling sequences
Lattice point enumeration
Root polytope
Staircase triangulation

ABSTRACT

We introduce an excedance statistic for the group of affine permutations $\widetilde{\mathfrak{S}}_n$ and determine the generating function of its distribution. The proof involves working with enumerating lattice points in a skew version of the root polytope of type A. We also show that the left coset representatives of the quotient $\widetilde{\mathfrak{S}}_n/\mathfrak{S}_n$ correspond to increasing juggling sequences and determine their Poincaré series.

1. Introduction

The symmetric group $\mathfrak{S}_n$ has many interesting permutation statistics. The most well-known statistics are inversions, descents, excedances, and the major index. The two most classical results are the descent statistic and the excedance statistic are equidistributed, and the inversion statistic and the major index are equidistributed. The symmetric group $\mathfrak{S}_n$ is also a finite Weyl group, which is a special case of Coxeter groups. In this terminology the group is known as A_{n-1} and it is viewed as the group generated by reflections in the hyperplanes $x_i = x_{i+1}$, $1 \leqslant i \leqslant n-1$. To every finite Weyl group W there is the associated affine Weyl group $\widetilde{W}$. Geometrically this is obtained by adding one more generator to the group corresponding to a reflection in an affine hyperplane which makes the group infinite. In the case of the symmetric group the affine hyperplane is $x_1 = x_n + 1$ and the group is denoted by $\widetilde{A}_{n-1}$.

* Corresponding author.
E-mail addresses: eclark@ms.uky.edu (E. Clark), jrge@ms.uky.edu (R. Ehrenborg).

0196-8858/$ – see front matter
doi:10.1016/j.aam.2009.12.006

Lusztig [11] found the following combinatorial description of the affine Weyl group $\widetilde{A}_{n-1}$. Define an affine permutation π to be a bijection $\pi : \mathbb{Z} \to \mathbb{Z}$ satisfying the following two conditions:

$$\pi(i+n) = \pi(i) + n \quad \text{for all } i, \quad \text{and} \quad \sum_{i=1}^{n} \bigl(\pi(i) - i\bigr) = 0. \tag{1.1}$$

Let $\widetilde{\mathfrak{S}}_n$ be the set of all affine permutations. It is straightforward that to see that $\widetilde{\mathfrak{S}}_n$ forms a group under composition. Lusztig then asserts that the group of affine permutations $\widetilde{\mathfrak{S}}_n$ and the affine Weyl group $\widetilde{A}_{n-1}$ are isomorphic. Furthermore, viewing $\widetilde{\mathfrak{S}}_n$ as a Coxeter group, it has n generators $s_1, \ldots, s_n$ given by

$$s_i(j) = \begin{cases} j+1 & \text{if } j \equiv i \bmod n, \\ j-1 & \text{if } j \equiv i+1 \bmod n, \\ j & \text{otherwise.} \end{cases}$$

Furthermore, the Coxeter relations when $n \geqslant 3$ are

$$s_i^2 = 1, \qquad (s_i s_{i+1})^3 = 1, \quad \text{and} \quad (s_i s_j)^2 = 1 \quad \text{for } i - j \not\equiv -1, 0, 1 \bmod n,$$

where we view the indices modulo n. For $n = 2$ the relations are $s_1^2 = s_2^2 = 1$ and there is no relation between s_1 and s_2. Observe that the symmetric group $\mathfrak{S}_n$ is embedded in the group of affine permutations. We can view the symmetric group as generated by the reflections $s_1, \ldots, s_{n-1}$.

Shi [15] and Björner and Brenti [4] were the first to study the group of affine permutations $\widetilde{\mathfrak{S}}_n$. Björner and Brenti extended the inversion statistic from the symmetric group $\mathfrak{S}_n$ to the group of affine permutations $\widetilde{\mathfrak{S}}_n$ by defining

$$\operatorname{inv}(\pi) = \sum_{1 \leqslant i < j \leqslant n} \left| \left\lfloor \frac{\pi(j) - \pi(i)}{n} \right\rfloor \right|, \quad \text{for } \pi \in \widetilde{\mathfrak{S}}_n. \tag{1.2}$$

Recall that the length of an element in a Coxeter group is given by the minimum number of generators required to express the group element as a product of generators. Shi showed that the inversion number is equal to the length of the affine permutation [15].

The next step is to look at the distribution of the inversions statistic, i.e., the length. Bott's formula solves this for any affine Weyl group in terms of the exponents of the finite group [5]. In the type A case one has

$$\sum_{\pi \in \widetilde{\mathfrak{S}}_n} q^{\ell(\pi)} = \frac{1 - q^n}{(1-q)^n}.$$

Björner and Brenti gave a combinatorial proof of this generating function identity by finding a bijection between $\widetilde{\mathfrak{S}}_n$ and $\mathbb{N}^n - \mathbb{P}^n$, that is, the collection of n-tuples with at least one zero. An earlier combinatorial proof was given by Ehrenborg and Readdy [9] using juggling sequences.

In this paper we will refine the Ehrenborg–Readdy juggling approach to give a proof of the length distribution of the coset representatives of the parabolic subgroup. We then extend the excedance statistic from the symmetric group $\mathfrak{S}_n$ to affine permutations. It is well known that the generating polynomial for the excedance statistic is given by the Eulerian polynomial, that is,

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\operatorname{exc}(\pi)} = \sum_{k=0}^{n-1} A(n, k+1) q^k,$$

where $A(n,k)$ is the Eulerian number. For affine permutations we determine the associated generating function for the excedance statistic. In doing so we reformulate the problem to instead counting lattice points on the hyperplane $x_1 + \cdots + x_n = 0$ which are certain distances in the ℓ^1-norm from the point $(-1, \ldots, -1, 0, \ldots, 0)$. The proof involves working with the $(n-1)$-dimensional root polytope R_{n-1} of type A defined as the convex hull of the vectors $\mathbf{v}_{i,j} = \mathbf{e}_i - \mathbf{e}_j$ for $1 \leqslant i, j \leqslant n$. The Ehrhart series for the root polytope is given by

$$\mathrm{Ehr}(R_{n-1}, t) = \frac{\sum_{i=0}^{n-1} \binom{n-1}{i}^2 t^i}{(1-t)^n}; \tag{1.3}$$

see [2,7,12,13].

Recently Ardila, Beck, Hosten, Pfeifle, and Seashore gave a natural triangulation of the root polytope and a combinatorial description of this triangulation [1]. Unfortunately the excedance statistic requires us to work with a skew version of the root polytope, but the triangulation still applies to the skew root polytope.

The faces of the Ardila–Beck–Hosten–Pfeifle–Seashore triangulation are in a natural correspondence with a combinatorial structure they named staircases. For a quick example of a staircase, see Fig. 2. Ardila et al. enumerated the number of staircases and used this to give a combinatorial proof of the Ehrhart series of the root polytope. In order to count lattice points inside the skew root polytope we use an additional parameter ℓ corresponding to an extra additive term in the dilation of the associated simplex in the triangulation of the skew root polytope; compare Figs. 4 and 5. This extra condition requires the associated staircase to have ℓ of the first k columns non-empty. Completing this enumeration allows us to count the lattice points inside the skew root polytope and determine the generating function for the affine excedance set statistics.

We end the paper with open questions and directions for further research.

2. Coset representatives and increasing juggling patterns

The affine symmetric group $\widetilde{\mathfrak{S}}_n$ consists of all bijections satisfying the relations in Eq. (1.1). This is a group under composition. From the first condition observe that the affine permutation π is uniquely determined by the entries $\pi(1), \pi(2), \ldots, \pi(n)$. Moreover, since π is a bijection, it permutes the congruency classes modulo n. Hence we can write

$$\pi(i) = nr_i + \sigma(i),$$

where r_i is an integer such that $\sum_{i=1}^n r_i = 0$ and $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation in $\mathfrak{S}_n$. Following Björner and Brenti [4] we write this as

$$\pi = (r_1, \ldots, r_n | \sigma) = (\mathbf{r} | \sigma).$$

Observe that the embedding of the symmetric group $\mathfrak{S}_n$ in the group of affine permutations is exactly the map which sends the permutation σ to the affine permutation $(\mathbf{0}|\sigma)$.

Consider a left coset D in the quotient $\widetilde{\mathfrak{S}}_n / \mathfrak{S}_n$. To pick a coset representative it is natural to choose the element π of least length in the coset D. This element π satisfies the inequalities $\pi(1) < \pi(2) < \cdots < \pi(n)$. We will study these coset representatives by considering their associated juggling sequences.

We refer the reader to the papers [6,8,9] and the book [14] for more on the mathematics of juggling. Here we give a brief introduction. A *juggling sequence* of period n is a sequence $\mathbf{a} = (a_1, \ldots, a_n)$ of positive integers such that $a_i + i$ are distinct modulo n. This can be viewed as a directed graph where there is an edge from t to $t + a_{t \bmod n}$ for all integers t. This symbolizes that a ball thrown at time t is caught at time $t + a_{t \bmod n}$. At each vertex of this graph the indegree and outdegree are

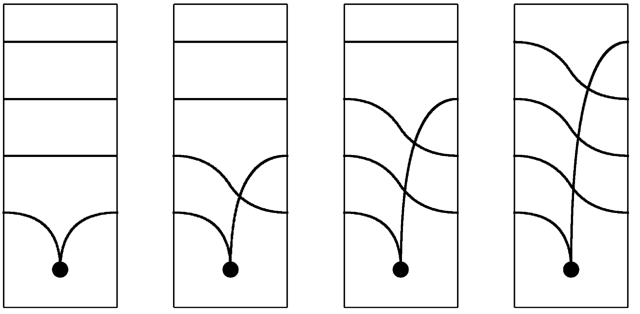

Fig. 1. The four juggling cards C_0^*, C_1^*, C_2^* and C_3^*.

each 1. This directed graph decomposes into connected components and each component is an infinite path. A path corresponds to a ball in the time–space continuum. The number of balls of is given by the mean value $(a_1 + \cdots + a_n)/n$; see [6].

A *crossing* is two directed edges $i \to j$ and $k \to \ell$ such that $i < k < j < \ell$; see [9]. The number of crossings $\mathrm{cross}(\mathbf{a})$ of a juggling sequence $\mathbf{a}$ is the number of crossings such that $1 \leqslant i \leqslant n$. This extra condition implies that number of crossings is finite and we are not counting crossings that are equivalent by a shift of a multiple of the period n.

We call a juggling sequence $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *increasing* if $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$. In juggling terms this states that a ball thrown at time i is caught before the ball thrown at time j, for $1 \leqslant i < j \leqslant n$.

Theorem 2.1. *The sum of $q^{\mathrm{cross}(\mathbf{a})}$ over all increasing juggling sequences $\mathbf{a}$ of period n having at most m balls is given by the Gaussian coefficient*

$$\sum_{\mathbf{a}} q^{\mathrm{cross}(\mathbf{a})} = \begin{bmatrix} m+n-1 \\ n \end{bmatrix}.$$

Similarly, the sum of $q^{\mathrm{cross}(\mathbf{a})}$ over all increasing juggling sequences $\mathbf{a}$ of period n having exactly m balls is given by the Gaussian coefficient

$$\sum_{\mathbf{a}} q^{\mathrm{cross}(\mathbf{a})} = q^{m-1} \begin{bmatrix} m+n-2 \\ n-1 \end{bmatrix}.$$

Proof. We prove this using juggling cards. See Fig. 1. Note that these juggling cards are the mirror images of the cards introduced in [9]. As in [9] by taking n juggling cards $C_{i_1}^*, C_{i_2}^*, \ldots, C_{i_n}^*$ and repeating them we construct a juggling pattern of period n having $\max(i_1, i_2, \ldots, i_n) + 1$ number of balls. However, note that with these cards it is always the ball in the lowest orbit that lands first. That is, the balls land according to their height.

If we use the cards $C_{i_1}^*, C_{i_2}^*, \ldots, C_{i_n}^*$, where $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n$, the ball thrown at time j will be in an orbit lower than the ball thrown at time $j+1$. Hence the ball thrown at time j will land before the next ball thrown. Hence the juggling pattern will be increasing.

Also observe that if $i_j > i_{j+1}$ then the jth ball would land after the $(j+1)$st ball and the pattern would not be increasing. Thus all increasing juggling sequences are in bijective correspondence with weakly increasing lists of indices.

Since the card C_i^* has i crossings, the sought after sum is given by

$$\sum_{\mathbf{a}} q^{\mathrm{cross}(\mathbf{a})} = \sum_{0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \leqslant m-1} q^{i_1 + i_2 + \cdots + i_n},$$

which is one of the combinatorial expressions for the Gaussian coefficient $\left[\begin{smallmatrix} m+n-1 \\ n \end{smallmatrix}\right]$.

To obtain the number of increasing juggling patterns having exactly m balls, we require the last card to be C^*_{m-1}. Thus the sum is restricted by the condition $i_n = m-1$, giving the factor q^{m-1}. The sum is now over $0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_{n-1} \leqslant m-1$ which produces the desired Gaussian coefficient. □

There is a natural bijection between juggling patterns **a** having exactly m balls and affine permutations π such that $i - \pi(i) < m$ for all i. Namely, given an affine permutation π, define the juggling pattern $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ by $a_i = m - i + \pi(i)$ for $i = 1, \ldots, n$. This states that the ball thrown at time i is caught at time $\pi(i) + m$. Furthermore, Theorem 4.2 in [9] states that the length of the affine permutation π and the number of crossings of the juggling pattern are related by

$$\ell(\pi) + \mathrm{cross}(\mathbf{a}) = n \cdot (m-1). \tag{2.1}$$

Theorem 2.2. *The sum $q^{\ell(\pi)}$ over all affine permutations $\pi \in \widetilde{\mathfrak{S}}_n$ such that $\pi(1) < \pi(2) < \cdots < \pi(n)$ is given by*

$$\sum_{\pi} q^{\ell(\pi)} = \frac{1}{(1-q)^{n-1}[n-1]!}.$$

Proof. Consider a coset representative π with the extra condition that $i - \pi(i) < m$ for all i. The condition $\pi(1) < \cdots < \pi(n)$ implies that π corresponds to an increasing juggling pattern having exactly m balls. By Eq. (2.1) and Theorem 2.1, we have

$$\sum_{\substack{\pi(1)<\cdots<\pi(n) \\ i-\pi(i)<m}} q^{n(m-1)-\ell(\pi)} = q^{m-1} \begin{bmatrix} m+n-2 \\ n-1 \end{bmatrix}.$$

By dividing by $q^{n(m-1)}$, substituting $q \mapsto q^{-1}$ and using the fact that Gaussian coefficients are symmetric, we obtain

$$\sum_{\substack{\pi(1)<\cdots<\pi(n) \\ i-\pi(i)<m}} q^{\ell(\pi)} = \begin{bmatrix} m+n-2 \\ n-1 \end{bmatrix} = \frac{[m+n-2][m+n-3]\cdots[m]}{[n-1]!}.$$

Finally by letting m tend to infinity the result follows. □

Observe that this gives another evaluation of the Poincaré series of the group of affine permutations.

Corollary 2.3. *The Poincaré series for $\widetilde{\mathfrak{S}}_n$ is given by*

$$\sum_{\pi \in \widetilde{\mathfrak{S}}_n} q^{\ell(\pi)} = \frac{1-q^n}{(1-q)^n}.$$

Proof. We have that

$$\begin{aligned} \sum_{\pi \in \widetilde{\mathfrak{S}}_n} q^{\ell(\pi)} &= \Bigg(\sum_{\pi \in \widetilde{\mathfrak{S}}_n/\mathfrak{S}_n} q^{\ell(\pi)} \Bigg) \Bigg(\sum_{\sigma \in \mathfrak{S}_n} q^{\ell(\sigma)} \Bigg) \\ &= \frac{[n]!}{(1-q)^{n-1}[n-1]!} \\ &= \frac{1-q^n}{(1-q)^n}. \end{aligned}$$

□

The approach in Theorems 2.1 and 2.2 presents a bijection between the coset representatives and partitions λ of length at most $n-1$. Such a bijection was given in [4, Theorem 4.4]. Given a partition $\lambda = (\lambda_1 \leqslant \cdots \leqslant \lambda_{n-1})$, let $(a_1, a_2, \ldots, a_n)$ be the increasing juggling sequence defined using the juggling cards $C_0^*, C_{\lambda_1}^*, \ldots, C_{\lambda_{n-1}}^*$. Note that $a_1 = 1$, that is, this juggling sequence begins with a 1 throw. Hence we cannot subtract a positive integer from each entry to make another juggling sequence. Let m be the number of balls of this juggling pattern, that is, $m = \lambda_{n-1} + 1$. Now construct the affine permutation by $\pi(i) = a_i + i - m$ for $1 \leqslant i \leqslant m$. The inverse of this bijection is given by letting $m = 2 - \pi(1)$ and $a_i = \pi(i) - i + m$. Then the partition is obtained by determining which juggling cards are used to create the juggling sequence $(a_1, \ldots, a_n)$.

This bijection differs from the one given by Björner and Brenti [4]. Their bijection has the extra advantage that the entries of the partition record inversions of the affine permutation.

3. Affine excedances

Recall that for a permutation $\sigma \in \mathfrak{S}_n$, an excedance of σ is an index i such that $i < \sigma(i)$. The *excedance statistic* of σ is the number of excedances, that is,

$$\operatorname{exc}(\sigma) = \bigl|\bigl\{i \in [n]:\ i < \sigma(i)\bigr\}\bigr|.$$

Observe that a permutation has at most $n-1$ excedances. The number of permutations in $\mathfrak{S}_n$ with k excedances is given by the Eulerian number $A(n, k+1)$.

Definition 3.1. For an affine permutation $\pi \in \widetilde{\mathfrak{S}}_n$ define the excedance statistic by

$$\operatorname{exc}(\pi) = \sum_{i=1}^{n} \left| \left\lceil \frac{\pi(i) - i}{n} \right\rceil \right|.$$

Observe that this definition of excedances agrees with the classical definition on the symmetric group, that is, for a permutation σ we have that $\operatorname{exc}((\mathbf{0}|\sigma)) = \operatorname{exc}(\sigma)$.

In order to analyze the distribution of the excedance statistic, we need to introduce a few notions. The ℓ^1-norm of a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is given by

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|.$$

The n-dimensional crosspolytope is given by $\Diamond_n = \operatorname{conv}\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_n\}$, where the $\mathbf{e}_i$ is the ith standard unit vector in $\mathbb{R}^n$. Note that $\partial\Diamond_n$ is the unit sphere in the ℓ^1-norm. Let H_n be the hyperplane in $\mathbb{R}^n$ defined by

$$H_n = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n:\ \sum_{i=1}^{n} x_i = 0 \right\},$$

and let L_n be the lattice $L_n = H_n \cap \mathbb{Z}^n$.

We now reformulate the excedance statistic of an affine permutation.

Proposition 3.2. *For $\sigma \in \mathfrak{S}_n$, define the vector $\mathbf{p}_\sigma \in \{-1, 0\}^n$ by $p_\sigma(i) = -1$ if i is an excedance of σ and 0 otherwise. Then for an affine permutation $\pi \in \widetilde{\mathfrak{S}}_n$ with $\pi = (r_1, \ldots, r_n|\sigma) = (\mathbf{r}|\sigma)$, the excedance statistic is given by*

$$\operatorname{exc}(\pi) = \|\mathbf{r} - \mathbf{p}_\sigma\|_1.$$

Proof. For $1 \leqslant i \leqslant n$ we have that

$$\left| \left\lceil \frac{\pi(i) - i}{n} \right\rceil \right| = \left| \left\lceil \frac{nr_i + \sigma(i) - i}{n} \right\rceil \right| = \begin{cases} |r_i| & \text{if } i \geqslant \sigma(i), \\ |r_i + 1| & \text{if } i < \sigma(i). \end{cases}$$

That is, we get this "+1" wherever we have an excedance in the permutation σ. The result follows by summing over all i. □

The next lemma expresses the generating function of affine excedances in terms of Eulerian numbers and generating functions of distances.

Lemma 3.3. *Let $\mathbf{p}_k$ be the lattice point $(\underbrace{-1, \ldots, -1}_{k}, 0, \ldots, 0)$ in $\mathbb{R}^n$. Then the following identity holds*

$$\sum_{\pi \in \widetilde{\mathfrak{S}}_n} q^{\mathrm{exc}(\pi)} = \sum_{k=0}^{n-1} A(n, k+1) \sum_{\mathbf{r} \in L_n} q^{\|\mathbf{r} - \mathbf{p}_k\|_1}.$$

Proof. Let σ be a permutation with k excedances. By permuting the coordinates of the vector $\mathbf{r}$, we have that

$$\sum_{\mathbf{r} \in L_n} q^{\|\mathbf{r} - \mathbf{p}_\sigma\|_1} = \sum_{\mathbf{r} \in L_n} q^{\|\mathbf{r} - \mathbf{p}_k\|_1}.$$

Since there are $A(n, k+1)$ permutations with k excedances, the lemma follows. □

This lemma reduces the problem of determining the number of affine permutations with i excedances to computing the number of points in the lattice L_n at distance i from the points $\mathbf{p}_0, \ldots, \mathbf{p}_{n-1}$. We begin by noting the following lemma.

Lemma 3.4. *For $\mathbf{r} \in L_n$ and $0 \leqslant k \leqslant n-1$ we have that $\|\mathbf{r} - \mathbf{p}_k\|_1 \geqslant k$ and $\|\mathbf{r} - \mathbf{p}_k\|_1 \equiv k \bmod 2$.*

Proof. For an integer x we have that $|x| \geqslant x$, hence

$$\|\mathbf{r} - \mathbf{p}_k\|_1 = \sum_{i=1}^{n} |r_i - p_i| \geqslant \sum_{i=1}^{n} (r_i - p_i) = k.$$

Also observe that $|x|$ and x have the same parity. That is $|x| \equiv x \bmod 2$. Therefore,

$$\|\mathbf{r} - \mathbf{p}_k\|_1 = \sum_{i=1}^{n} |r_i - p_i| \equiv \sum_{i=1}^{n} (r_i - p_i) = k \bmod 2. \qquad \square$$

This lemma tells us that for $0 \leqslant k \leqslant n-1$, the boundary of the crosspolytope centered at $\mathbf{p}_k$ will not intersect lattice points in L_n until its kth dilation and then only every other integer dilation after that. Thus we are interested in lattice points $\mathbf{r}$ at distance $2t + k$ from $\mathbf{p}_k$, where t is a non-negative integer. Therefore, we define the following polytope which will be the focus of our study.

Definition 3.5. For non-negative integers t and k, we define $P_{t,k}$ to be the set

$$P_{t,k} = \big((2t+k)\Diamond_n + \mathbf{p}_k\big) \cap H_n.$$

That is, $\mathbf{x} \in P_{t,k}$ if and only if $\|\mathbf{x} - \mathbf{p}_k\|_1 \leqslant 2t + k$ and $\mathbf{x} \in H_n$.

In the case $t = k = 0$ we have that $P_{0,0}$ is a point. In the other cases, $P_{t,k}$ is obtained by cutting a dilated crosspolytope with a hyperplane which is parallel to two facets of the crosspolytope. For $k \geqslant 1$ and $t = 0$ the hyperplane is the affine span of a facet of the crosspolytope and hence the set $P_{0,k}$ is an $(n-1)$-dimensional simplex. Finally, for $t > 0$ the hyperplane cuts the interior of crosspolytope. Hence the combinatorial type of $P_{t,k}$ in this case does not depend on the parameters t and k.

4. The root polytope

We begin to study the case $t = 1$ and $k = 0$. That is, we are intersecting the crosspolytope $2\Diamond_n$ with the hyperplane H_n. This is the root polytope and its structure will be used in developing the other cases. We first verify that $P_{1,0}$ is the root polytope.

Proposition 4.1. *The polytope $P_{1,0}$ is the $(n-1)$-dimensional root polytope R_{n-1}, that is, its vertices are $\mathbf{v}_{i,j} = \mathbf{e}_i - \mathbf{e}_j$ for $1 \leqslant i, j \leqslant n$ and $i \neq j$.*

Proof. The vertices of the crosspolytope $2\Diamond_n$ are partitioned into the two sets $\{2\mathbf{e}_i\}_{1\leqslant i\leqslant n}$ and $\{-2\mathbf{e}_i\}_{1\leqslant i\leqslant n}$ by the hyperplane H_n. Hence the edges of the crosspolytope that are cut by H_n are of the form $[2\mathbf{e}_i, -2\mathbf{e}_j]$ for $i \neq j$. The midpoint of these edges is $\mathbf{e}_i - \mathbf{e}_j = \mathbf{v}_{i,j}$, which are precisely the vertices of $P_{1,0}$. $\square$

We introduce now the work of Ardila, Beck, Hosten, Pfeifle, and Seashore [1] who have studied the combinatorial structure of the root polytope in depth. The next definition and theorem are due to them.

Definition 4.2. (Ardila et al.) We call the list $I = ((i_1, j_1), (i_2, j_2), \ldots, (i_m, j_m))$ a *staircase* of size m in an n by n array if

1. $1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_m \leqslant n$ and $1 \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_m \leqslant n$,
2. $(i_s, j_s) \neq (i_t, j_t)$ for $s \neq t$, and
3. $i_s \neq j_t$, $1 \leqslant s, t \leqslant m$.

Let $\mathbf{v}_I$ denote the set $\mathbf{v}_I = \{\mathbf{v}_{i_1,j_1}, \mathbf{v}_{i_2,j_2}, \ldots, \mathbf{v}_{i_m,j_m}\}$.

Note that the third condition above is the essential condition. One of its implications is that the diagonal element (i,i) is not part of any staircase. In pictures we always shade these diagonal elements. Also note that if we remove a pair from a staircase, the resulting list is also a staircase. That is, the collection of staircases forms a simplicial complex. This simplicial complex is in fact spherical.

Theorem 4.3. *(Ardila et al.) The collection $\{\mathrm{conv}(\mathbf{v}_I)\}_I$, where I ranges over all staircases in an n by n array, is a triangulation of the boundary of the root polytope R_{n-1}, that is, ∂R_{n-1}.*

The three-dimensional root polytope R_3 is the cuboctahedron, which consists of 8 triangles and 6 squares. However, in the triangulation of its boundary, each square is cut into 2 triangles. Hence the triangulation in this case has $8 + 2 \cdot 6 = 20$ facets. The staircase in Fig. 2 corresponds to an edge of the cuboctahedron. This edge lies in two facets: the facet $\mathrm{conv}(\mathbf{v}_{1,2}, \mathbf{v}_{3,2}, \mathbf{v}_{4,2})$, which is a triangular face of the cuboctahedron, and the facet $\mathrm{conv}(\mathbf{v}_{1,2}, \mathbf{v}_{3,2}, \mathbf{v}_{3,4})$, which is one-half of the square face $\mathrm{conv}(\mathbf{v}_{1,2}, \mathbf{v}_{3,2}, \mathbf{v}_{3,4}, \mathbf{v}_{1,4})$.

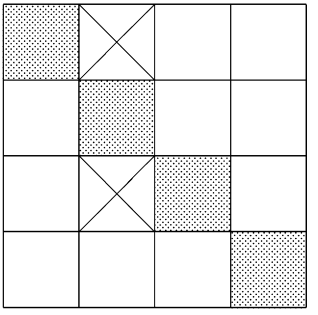

Fig. 2. A visualization of the staircase $I = ((1,2),(3,2))$. Observe that this staircase is only contained in two other staircases, namely by adding either $(4,2)$ or $(3,4)$.

Definition 4.4. For a staircase $I = ((i_1, j_1), (i_2, j_2), \dots, (i_m, j_m))$ let Γ_I be the m-dimensional simplex $\mathrm{conv}(\{\mathbf{0}\} \cup \mathbf{v}_I)$. Furthermore, let C_I be the simplicial cone generated by the set $\mathbf{v}_I$, that is,

$$C_I = \left\{ \sum_{s=1}^{m} \lambda_s \mathbf{v}_{i_s, j_s} \colon \lambda_s \geqslant 0 \right\}.$$

Theorem 4.3 implies that $\{C_I \colon I \text{ is a staircase}\}$ is a complete simplicial fan. In particular, we know that the hyperplane H_n is the disjoint union of the relative interiors of the cones C_I, that is,

$$H_n = \biguplus_I \mathrm{relint}(C_I),$$

where I ranges over all staircases. Thus, for every lattice point $\mathbf{w} \in L_n$ we know that $\mathbf{w}$ is contained in the relative interior of one cone C_I for exactly one staircase $I = ((i_1, j_1), \dots, (i_m, j_m))$. In that case we may write

$$\mathbf{w} = \sum_{s=1}^{m} \lambda_s \mathbf{v}_{i_s, j_s},$$

where each λ_s is a positive integer.

5. The skew root polytope

We now investigate the number of lattice points contained in the polytope $P_{t,k}$.

Proposition 5.1. *Let $\mathbf{w}$ be a lattice point in the relative interior of the cone C_I, that is, $\mathbf{w} \in L_n \cap \mathrm{relint}(C_I)$, where I is the staircase $((i_1, j_1), \dots, (i_m, j_m))$. In other words, we can write $\mathbf{w}$ as the positive linear combination $\mathbf{w} = \sum_{s=1}^{m} \lambda_s \mathbf{v}_{i_s, j_s}$. Then the ℓ^1-norm between the two points $\mathbf{w}$ and $\mathbf{p}_k$ is given by*

$$\|\mathbf{w} - \mathbf{p}_k\|_1 = k - 2\ell + 2 \sum_{s=1}^{m} \lambda_s,$$

where $\ell = |[k] \cap \{j_1, j_2, \dots, j_m\}|$, that is, ℓ is the number of non-empty columns among the first k columns of the staircase I.

Proof. Consider the difference

$$\begin{aligned}\mathbf{w}-\mathbf{p}_k &= \sum_{s=1}^{m}\lambda_s(\mathbf{e}_{i_s}-\mathbf{e}_{j_s})+\sum_{s=1}^{k}\mathbf{e}_s\\ &= \sum_{r=1}^{k}\left(\begin{cases}\lambda_s+1 & \text{if } r=i_s,\\ -\lambda_s+1 & \text{if } r=j_s,\\ 1 & \text{otherwise}\end{cases}\right)\cdot\mathbf{e}_r+\sum_{r=k+1}^{n}\left(\begin{cases}\lambda_s & \text{if } r=i_s,\\ -\lambda_s & \text{if } r=j_s,\\ 0 & \text{otherwise}\end{cases}\right)\cdot\mathbf{e}_r.\end{aligned}$$

Using that $\lambda_s \geqslant 1$ we have that the ℓ^1-norm is given by

$$\|\mathbf{w}-\mathbf{p}_k\|_1=\sum_{r=1}^{k}\begin{cases}\lambda_s+1 & \text{if } r=i_s,\\ \lambda_s-1 & \text{if } r=j_s,\\ 1 & \text{otherwise}\end{cases}+\sum_{r=k+1}^{n}\begin{cases}\lambda_s & \text{if } r=i_s,\\ \lambda_s & \text{if } r=j_s,\\ 0 & \text{otherwise.}\end{cases}$$

Each λ_s appears twice in this sum. Furthermore, there are $k-\ell$ ones and ℓ negative ones also in the sum. This proves the proposition. □

Definition 5.2. For a staircase $I=((i_1,j_1),\dots,(i_m,j_m))$ define the apex $\mathbf{a}_I$ to be the sum $\mathbf{a}_I=\sum_{s=1}^{m}\mathbf{v}_{i_s,j_s}$. Furthermore, let $\Delta_{t,k}(I)$ be the simplex

$$\Delta_{t,k}(I)=\mathbf{a}_I+(t-m+\ell)\Gamma_I,$$

where ℓ is defined as in Proposition 5.1.

Theorem 5.3. *The collection of simplices $\{\Delta_{t,k}(I)\}_I$, where I ranges over all staircases, partitions the lattice points of $P_{t,k}$ into disjoint sets.*

Proof. Observe that the simplex $\Delta_{t,k}(I)$ is contained in the relative interior of C_I and that the relative interiors are pairwise disjoint. Hence the simplices $\{\Delta_{t,k}(I)\}_I$ are pairwise disjoint. Now assume that $\mathbf{w}$ is a lattice point inside the polytope $P_{t,k}$. Then there is exactly one staircase $I=((i_1,j_1),\dots,(i_m,j_m))$ such that $\mathbf{w}=\sum_{s=1}^{m}\lambda_s\mathbf{v}_{i_s,j_s}$, where each λ_s is a positive integer. By Proposition 5.1 and the definition of the polytope $P_{t,k}$, we have that

$$k-2\ell+2\sum_{s=1}^{m}\lambda_s=\|\mathbf{w}-\mathbf{p}_k\|_1\leqslant 2t+k.$$

By cancelling k on both sides, dividing by 2 and subtracting m, we have

$$\sum_{s=1}^{m}(\lambda_s-1)\leqslant t+\ell-m. \tag{5.1}$$

We can now write $\mathbf{w}$ as

$$\mathbf{w}=\sum_{s=1}^{m}\lambda_s\mathbf{v}_{i_s,j_s}=\mathbf{a}_I+\sum_{s=1}^{m}(\lambda_s-1)\mathbf{v}_{i_s,j_s}.$$

Now by the inequality (5.1) we have $\mathbf{w}\in\mathbf{a}_I+(t+\ell-m)\Gamma_I=\Delta_{t,k}(I)$, proving that each lattice point is inside at least one simplex $\Delta_{t,k}(I)$. □

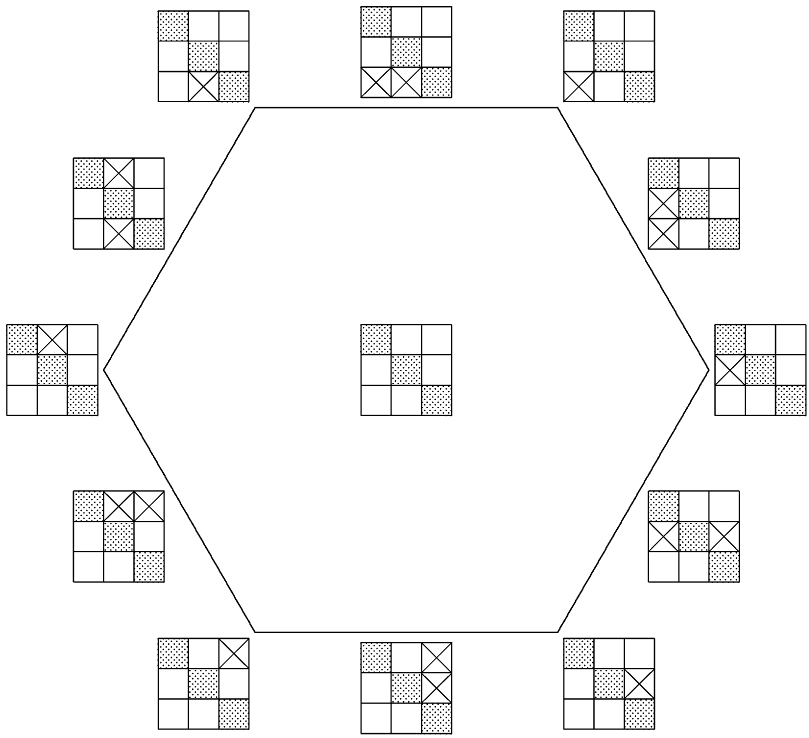

Fig. 3. The root polytope $P_{1,0}$ (hexagon) with faces labeled with the associated staircases.

Proposition 5.4. *Let I be a staircase of size m and let ℓ be the number of non-empty columns among the first k columns. Then the number of lattice points contained in the simplex $\Delta_{t,k}(I)$ and the number of lattice points in the simplex $\Delta_{t,k}(I)$ intersected with the boundary of $P_{t,k}$ are given by*

$$\left|\Delta_{t,k}(I) \cap L_n\right| = \binom{t+\ell}{m},$$
$$\left|\Delta_{t,k}(I) \cap \partial P_{t,k} \cap L_n\right| = \binom{t+\ell-1}{m-1}.$$

Proof. We know by definition that $\Delta_{t,k}(I)$ is the $(t-m+\ell)$-dilation of a standard m-simplex. It is well known that the number of lattice points contained in this dilation is $\binom{m+(t-m+\ell)}{m} = \binom{t+\ell}{m}$. Furthermore, for a lattice point $\mathbf{w}$ to be on the boundary of $P_{t,k}$, there is equality in inequality (5.1). Hence $\Delta_{t,k}(I) \cap \partial P_{t,k}$ is the $(t-m+\ell)$-dilation of a standard $(m-1)$-simplex and the result follows. □

We are using the convention that $\binom{n}{-1} = \delta_{n,-1}$, so that Proposition 5.4 also holds for the empty staircase. Note that this convention agrees with the formal power series $\sum_{t\geqslant 0}\binom{t+m}{m}x^t = \frac{1}{(1-x)^{m+1}}$.

Combining Theorem 5.3 and Proposition 5.4 we have the following result.

Proposition 5.5. *The number of lattice points in the polytope $P_{t,k}$ and on its boundary $\partial P_{t,k}$ are given by*

$$\sum_I \binom{t+\ell}{m}, \quad \textit{respectively,} \quad \sum_I \binom{t+\ell-1}{m-1},$$

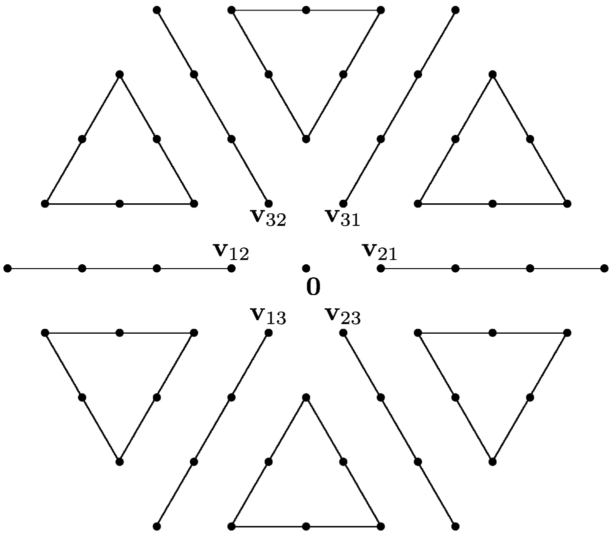

Fig. 4. Partitioning the lattice points of $P_{t,0}$. The origin corresponds to the empty face, the six lines to the vertices, and the six triangles to the edges. Hence the number of lattice points in $P_{t,0}$ is $\binom{t}{0} + 6\binom{t}{1} + 6\binom{t}{2}$.

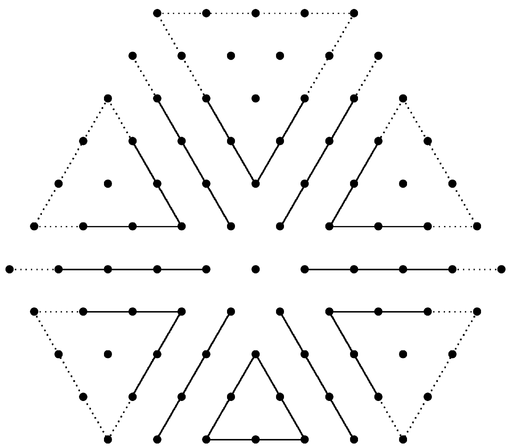

Fig. 5. Partitioning the lattice points of $P_{t,2}$. Observe that the number of lattice points are $\binom{t}{0} + 2 \cdot \binom{t}{1} + 4 \cdot \binom{t+1}{1} + \binom{t}{2} + 4 \cdot \binom{t+1}{2} + \binom{t+2}{2}$.

where I ranges over all staircases in an n by n array, m is the size of the staircase I and ℓ is the number of non-empty columns among the first k columns of I.

Example 5.6. We can visualize Proposition 5.5 as follows. Consider the case when $n = 3$, that is, the associated root polytope is a hexagon; see Fig. 3. First we view the case $k = 0$. The partitioning of the lattice points in the polytope $P_{t,0}$ is shown in Fig. 4. In this case each simplex $\Delta_{t,0}(I)$ contains $\binom{t}{m}$ lattice points. Next consider the case $k = 2$; see Fig. 5. Going from the simplices $\Delta_{t,0}(I)$ to the simplices $\Delta_{t,2}(I)$ observe that some of them have been stretched by an additive term of ℓ. By comparing the stretching factor with the labels in Fig. 3, we note that this additive term ℓ is exactly the number of non-empty columns among the first $k = 2$ columns of the staircase diagram.

6. Enumerating staircases

Proposition 6.1. *The number of staircases of size m in an n by n array with ℓ of the first k columns non-empty is given by*

$$\binom{k}{\ell}\binom{n-1}{m}\binom{n+m-k-1}{m-\ell}.$$

Proof. We begin by assuming a staircase will go through exactly a rows and b columns in the n by n array. We must first choose the ℓ of the first k columns that will be used in $\binom{k}{\ell}$ ways. We pick the remaining columns in $\binom{n-k}{a-\ell}$ ways. Now picking the disjoint rows can be done in $\binom{n-a}{b}$ ways. We have to get a path from the upper left to the lower right of this a by b subarray that goes through every row and column of the subarray. We must do this in $m-1$ steps where the steps are horizontal $(1,0)$, vertical $(0,1)$, and diagonal $(1,1)$. Note that we have to cover a vertical distance of $a-1$ and a horizontal distance of $b-1$. This can only be done with $m-a$ horizontal steps, $m-b$ vertical steps and $a+b-m-1$ diagonal steps. Hence the number of possibilities is given by the trinomial coefficient $\binom{m-1}{m-a,m-b,a+b-m-1}$. Thus, the number of paths is given by

$$\sum_{a,b}\binom{k}{\ell}\binom{n-k}{a-\ell}\binom{n-a}{b}\binom{m-1}{m-a,m-b,a+b-m-1}.$$

To evaluate this sum, consider

$$\begin{aligned}
&\binom{n-1}{\ell-1}\binom{n-\ell}{n-k}\sum_{a,b}\binom{n-k}{a-\ell}\binom{n-a}{b}\binom{m-1}{m-a,m-b,a+b-m-1}\\
&=\binom{n-1}{m}\binom{m-1}{\ell-1}\sum_{a}\binom{m-\ell}{a-\ell}\binom{n-a}{k-\ell}\sum_{b}\binom{m}{b}\binom{n-m-1}{n-a-b}\\
&=\binom{n-1}{m}\binom{m-1}{\ell-1}\sum_{a}\binom{m-\ell}{a-\ell}\binom{n-a}{k-\ell}\binom{n-1}{n-a}\\
&=\binom{n-1}{m}\binom{m-1}{\ell-1}\binom{n-1}{k-\ell}\sum_{a}\binom{m-\ell}{a-\ell}\binom{n+\ell-k-1}{n-a-k+\ell}\\
&=\binom{n-1}{m}\binom{m-1}{\ell-1}\binom{n-1}{k-\ell}\binom{m+n-k-1}{n-k}\\
&=\binom{n-1}{\ell-1}\binom{n-\ell}{n-k}\binom{n-1}{m}\binom{m+n-k-1}{m-\ell}.
\end{aligned}$$

The Vandermonde identity was used in the second and fourth steps. The other three steps are a veritable orgy of expressing the binomial coefficients in terms of factorials and shuffling the factorials around. The result now follows by multiplying by $\binom{k}{\ell}$ and dividing by $\binom{n-1}{\ell-1}$ and $\binom{n-\ell}{n-k}$. □

Now, combining Propositions 5.5 and 6.1, we immediately obtain the following theorem.

Theorem 6.2. *The number of lattice points contained in the polytope $P_{t,k}$ and on its boundary $\partial P_{t,k}$ is*

$$|P_{t,k}\cap L_n| = \sum_{m=\ell}^{n-1}\sum_{\ell=0}^{k}\binom{k}{\ell}\binom{n-1}{m}\binom{n+m-k-1}{m-\ell}\binom{t+\ell}{m},$$

respectively,

$$|\partial P_{t,k} \cap L_n| = \sum_{m=\ell}^{n-1} \sum_{\ell=0}^{k} \binom{k}{\ell} \binom{n-1}{m} \binom{n+m-k-1}{m-\ell} \binom{t+\ell-1}{m-1}.$$

Lemma 6.3. *For non-negative integers n and k,*

$$\sum_{\ell=0}^{k} \sum_{m=\ell}^{n-1} \binom{k}{\ell} \binom{n-1}{m} \binom{n+m-k-1}{m-\ell} x^{m-\ell} (1-x)^{n-m-1} = \sum_{i=0}^{n-1-k} \binom{n-1-k}{i} \binom{n-1+k}{n-1-i} x^i.$$

Proof. We start with

$$\begin{aligned}
&\sum_{\ell=0}^{k} \sum_{m=\ell}^{n-1} \binom{k}{\ell} \binom{n-1}{m} \binom{n+m-k-1}{m-\ell} x^{m-\ell} (1-x)^{n-m-1} \\
&= \sum_{\ell=0}^{k} \sum_{p=0}^{n-1-\ell} \binom{k}{\ell} \binom{n-1}{p} \binom{2n-2-k-p}{n-1-k+\ell} x^{n-1-\ell-p} (1-x)^p \\
&= \sum_{\ell=0}^{k} \sum_{p=0}^{n-1-\ell} \binom{k}{\ell} \binom{n-1}{p} \sum_{i=0}^{n-1-k} \binom{n-1-k}{i} \binom{n-1-p}{n-1-k+\ell-i} x^{n-1-\ell-p} (1-x)^p \\
&= \sum_{i=0}^{n-1-k} \binom{n-1-k}{i} x^{n-1-k-i} \sum_{\ell=0}^{k} \binom{k}{\ell} \binom{n-1}{i+k-\ell} \sum_{p=0}^{i+k-\ell} \binom{i+k-\ell}{p} (1-x)^p x^{i+k-\ell-p} \\
&= \sum_{i=0}^{n-1-k} \binom{n-1-k}{i} \binom{n-1+k}{i+k} x^{n-1-k-i} \\
&= \sum_{i=0}^{n-1-k} \binom{n-1-k}{i} \binom{n-1+k}{n-1-i} x^i.
\end{aligned}$$

In the first equality we make the substitution $p = n - m - 1$. The second equality comes from expanding the term $\binom{2n-2-k-p}{n-1-k+\ell}$ using the classical Vandermonde identity. The third equality is the trinomial coefficient identity

$$\begin{aligned}
\binom{n-1}{p} \binom{n-1-p}{n-1-k+\ell-i} &= \binom{n-1}{p, n-1-k+\ell-i, -p+k-\ell+i} \\
&= \binom{n-1}{n-1-k+\ell-i} \binom{k-\ell+i}{p} \\
&= \binom{n-1}{i+k-\ell} \binom{i+k-\ell}{p}.
\end{aligned}$$

Also note that the last binomial coefficient is zero for $i + k - \ell < p \leqslant n - 1 - \ell$. The fourth equality is the binomial theorem applied to $((1-x) + x)^{i+k-\ell} = 1$ followed by collapsing the sum over ℓ using the Vandermonde identity. The last step is by the substitution $i \mapsto n - 1 - k - i$ and by the symmetry of the binomial coefficients. $\square$

Proposition 6.4. *For any* $0 \leqslant k \leqslant n-1$,

$$\sum_{\mathbf{r}\in L_n} q^{\|\mathbf{r}-\mathbf{p}_k\|} = \frac{1}{(1-q^2)^{n-1}} \sum_{i=0}^{n-1-k} \binom{n-1-k}{i}\binom{n-1+k}{n-1-i} q^{2i+k}.$$

Proof. First observe that by the substitution $t = s + m - \ell$,

$$\begin{aligned}\sum_{t\geqslant 0}\binom{t+\ell-1}{m-1} q^{2t+k} &= q^{2(m-\ell)+k}\sum_{s\geqslant \ell-m}\binom{s+m-1}{m-1} q^{2s}\\ &= \frac{q^k}{(1-q^2)^{n-1}} q^{2(m-\ell)}\left(1-q^2\right)^{n-1-m}. \qquad (6.1)\end{aligned}$$

Hence we have that

$$\begin{aligned}\sum_{\mathbf{r}\in L_n} q^{\|\mathbf{r}-\mathbf{p}_k\|} &= \sum_{t\geqslant 0} |\partial P_{t,k}\cap L_n| q^{2t+k}\\ &= \sum_{t\geqslant 0}\sum_{m=\ell}^{n-1}\sum_{\ell=0}^{k}\binom{k}{\ell}\binom{n-1}{m}\binom{n+m-k-1}{m-\ell}\binom{t+\ell-1}{m-1} q^{2t+k}\\ &= \frac{q^k}{(1-q^2)^{n-1}}\sum_{m=\ell}^{n-1}\sum_{\ell=0}^{k}\binom{k}{\ell}\binom{n-1}{m}\binom{n+m-k-1}{m-\ell} q^{2(m-\ell)}\left(1-q^2\right)^{n-1-m}\\ &= \frac{q^k}{(1-q^2)^{n-1}}\sum_{i=0}^{n-1-k}\binom{n-1-k}{i}\binom{n-1+k}{n-1-i} q^{2i},\end{aligned}$$

where the third step is by Eq. (6.1) and the last step is Lemma 6.3. □

Observe that Proposition 6.4 reduces to the Ehrhart series of the root polytope R_{n-1} in the case $k = 0$; see Eq. (1.3). The difference in the power of $1 - t = 1 - q^2$ comes from Proposition 6.4 counting lattice points on the boundary, while the Ehrhart series counts lattice points in the polytope.

Finally, combining Lemma 3.3 with Proposition 6.4, we obtain the generating function associated with the excedance statistic of affine permutations.

Theorem 6.5. *The generating function for affine excedances is given by*

$$\sum_{\pi\in\widetilde{\mathfrak{S}}_n} q^{\mathrm{exc}(\pi)} = \frac{1}{(1-q^2)^{n-1}}\sum_{k=0}^{n-1} A(n,k+1)\sum_{i=0}^{n-1-k}\binom{n-1-k}{i}\binom{n-1+k}{n-1-i} q^{2i+k}.$$

7. Concluding remarks

There is not much known about how classic permutation statistics generalize to affine permutations. In analogy with the definitions of the inversion and excedance statistics (Eq. (1.2) and Definition 3.1), it is natural to consider the expression

$$f(\pi) = \sum_{i=1}^{n}\left|\left\lfloor \frac{\pi(i+1)-\pi(i)}{n}\right\rfloor\right| \qquad (7.1)$$

as an affine analogue of the descent statistic though it does not exactly generalize the descent statistic. However, there are only a finite number of affine permutations with a given value of the f statistic. Hence the generating function

$$\sum_{\pi \in \widetilde{\mathfrak{S}}_n} q^{f(\pi)}$$

is well defined.

The numerator in Proposition 6.4 has a nice combinatorial interpretation. The coefficient of q^{2i+k} counts the number of lattice paths from $(0,0)$ to $(n-1,n-1)$ which go through the point $(i, n-1-k-i)$. In particular, the sum of these coefficients is $\binom{2n-2}{n-1}$. Is there a more bijective reason for this interpretation?

Observe that the numerator in Proposition 6.4 is symmetric. Recall that the Ehrhart series of reflexive polytopes have this property and the root polytope is reflexive. Are there other reflexive polytopes that have a skew version with a symmetric numerator?

Proposition 6.1 enumerates staircases where ℓ of the first k columns are non-empty. The result is a product of three binomial coefficients. Is there a more bijective proof, which avoids all the binomial coefficient manipulations?

Since simple juggling patterns are so closely related with the affine Weyl group $\widetilde{A}_{n-1}$, it is natural to ask for juggling interpretations for the other Weyl groups. See the paper [10] for permutation interpretations for these groups.

Finally, is there an excedance statistic for finite Coxeter groups in general? Bagno and Garber [3] have extended the excedance statistic to the infinite classes B_n and D_n. Furthermore, could the statistic be extended to the associated affine groups? A first step in this direction would be to consider the B_n case, that is, the group of signed permutations, and see if their excedance statistic can be extended to the affine group $\widetilde{B}_n$. Would the associated calculations involve a skew version of the root polytope of type B?

Acknowledgments

The authors thank Serkan Hosten for introducing us to the reference [1] and our referee for his comments. Special thanks to Margaret Readdy and Ruriko Yoshida who read earlier versions of this paper. The second author was supported by National Science Foundation grant DMS 0902063.

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