The Hankel determinant of exponential polynomials*

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The Hankel matrix of order $n+1$ of a sequence $a_0, a_1, \ldots$ is the $n+1$ by $n+1$ matrix whose (i,j) entry is a_{i+j}, where the indices range between 0 and n. The *Hankel determinant* of order $n+1$ is the determinant of the corresponding Hankel matrix, that is,

$$\det\left(a_{i+j}\right)_{0 \leq i,j \leq n} = \det \begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{pmatrix}.$$

The purpose of this note is two-fold. First we present evaluations of Hankel determinants of sequences of combinatorial interest related to partitions and permutations. Many such computations have been carried out by Radoux in his sequence of papers [2]–[5]. His proof methods include using a functional identity due to Sylvester and factoring the Hankel matrix. Second, unlike Radoux, we instead give bijective proofs to reveal the underlying structure of these identities.

A partition $\pi = \{B_1, \ldots, B_k\}$ of a finite set S is a collection of non-empty subsets $B_1, \ldots, B_k$, called *blocks*, such that the blocks are disjoint and their union is the set S. Let $|\pi|$ denote the number of blocks in the partition π. The *exponential polynomials* $e_n(x)$ are defined by

$$e_n(x) = \sum_{\pi} x^{|\pi|},$$

where π ranges over all partitions of an n-element set. A few properties of the exponential polynomials are (i) $e_n(1)$ is equal to the nth Bell number, (ii) $e_n(x) = \sum_{k=0}^{n} S(n,k)x^k$ where $S(n,k)$ is the Stirling number of the second kind, (iii) $e_n(x) = e^{-x}\left(x \cdot d/dx\right)^n e^x$, (iv) $\sum_{n \geq 0} e_n(x)t^n/n! = \exp(x(e^t - 1))$. For more on the properties of exponential polynomials, see [7, Section 13], which is [6, pp. 7–82].

Theorem 1 (Radoux [2]) *The Hankel determinant of order $n+1$ of the exponential polynomials $e_n(x)$ is given by*

$$\det\left(e_{i+j}(x)\right)_{0 \leq i,j \leq n} = x^{(n+1)n/2} \cdot \prod_{i=0}^{n} i!.$$

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Setting $x = 1$ in the identity gives the Hankel determinant of the Bell numbers. This case was mentioned by Martin Gardner in [1, p. 35].

Let $R_0, \ldots, R_n, C_0, \ldots, C_n$ be disjoint sets such that the cardinality of R_i and C_i is i. We call R_i the ith *row set* and C_i the ith *column set*. Moreover, assume that the set R_i possesses a linear order. Let S be the disjoint union of all the row and column sets. A partition π of the set S and a permutation σ of the set $\{0, 1, \ldots, n\}$ are called *compatible* if for all blocks B in π there exists an index i such that $B \subseteq R_i \cup C_{\sigma(i)}$. Let T denote the set of all pairs (σ, π), where σ is a permutation and π is a compatible partition. Finally, let $(-1)^\sigma$ denote the sign of the permutation σ.

Proposition 2 *The Hankel determinant of the exponential polynomials $e_n(x)$ has the following combinatorial interpretation:*

$$\det\left(e_{i+j}(x)\right)_{0 \leq i,j \leq n} = \sum_{(\sigma,\pi) \in T} (-1)^\sigma \cdot x^{|\pi|}.$$

Proof: Consider the exponential polynomial $e_{i+j}(x)$ at row i and column j as a sum over all partitions of the set $R_i \cup C_j$. Then the determinant can be expanded as

$$\sum_\sigma (-1)^\sigma \cdot \left(\sum_{\pi_0} x^{|\pi_0|}\right) \cdots \left(\sum_{\pi_n} x^{|\pi_n|}\right),$$

where π_i is a partition on the set $R_i \cup C_{\sigma(i)}$. By letting $\pi = \pi_0 \cup \cdots \cup \pi_n$, we obtain a partition on the set S that is compatible with the permutation σ. Moreover, any pair $(\sigma, \pi) \in T$ may be obtained in this way. Finally, observe that $|\pi| = |\pi_0| + \cdots + |\pi_n|$. $\square$

Proof of Theorem 1: Let T_1 be the subset of T consisting of the pairs (σ, π), where σ is the identity permutation and each block B of π contains two elements, one from the row set R_i and one from the corresponding column set C_i. Observe that

$$\sum_{(\sigma,\pi) \in T_1} (-1)^\sigma \cdot x^{|\pi|} = x^{(n+1)n/2} \cdot \prod_{i=0}^{n} i!,$$

since π consists of $(n+1)n/2$ blocks and there are $i!$ ways to choose a matching between R_i and C_i.

Let $T_2 = T - T_1$. We show that there exists an involution on the set T_2 that changes the sign of the permutation and keeps the number of blocks of the partition the same. Such an involution implies that

$$\sum_{(\sigma,\pi) \in T_2} (-1)^\sigma \cdot x^{|\pi|} = 0,$$

and hence proves the theorem.

For (σ, π) in T define a_i to be the number of blocks B that intersect both R_i and $C_{\sigma(i)}$. Observe that $a_i \leq |R_i| = i$ and $a_i \leq |C_{\sigma(i)}| = \sigma(i)$. We call the a_i's the *crossing numbers.*

Consider first the case when all the crossing numbers a_i are distinct. Then we have $a_i = i$ for all indices i and we conclude that σ is the identity permutation. Moreover, π provides a matching between the sets R_i and C_i. Hence this implies that (σ, π) belongs to T_1.

Thus for (σ, π) in T_2 we know that there exists a pair of indices (j, k) such that $a_j = a_k$. Let (j, k) be the least such pair in the lexicographic order. Let σ' be the permutation $\sigma'(j) = \sigma(k)$, $\sigma'(k) = \sigma(j)$ and $\sigma'(i) = \sigma(i)$ for $i \neq j, k$. Clearly, $(-1)^{\sigma'} = -(-1)^{\sigma}$.

Now we construct a partition π' compatible with σ'. Let $a = a_j = a_k$. Let $B_1, \ldots, B_a$ be the blocks of $R_j \cup C_{\sigma(j)}$ that intersect both R_j and $C_{\sigma(j)}$. We can order these blocks in a canonical fashion according to the smallest element in the block intersected with R_j. This is why we assumed a linear order on the row set R_i. Similarly, let $D_1, \ldots, D_a$ be the blocks of $R_k \cup C_{\sigma(k)}$ that intersect both the row and the column set. Order these blocks in the same canonical fashion. Define new blocks by

$$B'_i = (B_i \cap R_j) \cup (D_i \cap C_{\sigma(k)}), \qquad D'_i = (D_i \cap R_k) \cup (B_i \cap C_{\sigma(j)}).$$

Let π' be the partition obtained from the partition π by replacing the blocks $B_1, \ldots, B_a, D_1, \ldots, D_a$ with $B'_1, \ldots, B'_a, D'_1, \ldots, D'_a$. Observe that the two partitions π and π' have the same number of blocks and π' is compatible with σ'.

We claim that the map $(\sigma, \pi) \longmapsto (\sigma', \pi')$ is an involution. Observe that both pairs (σ, π) and (σ', π') have the same sequence of crossing numbers. Hence the map, when applied again, chooses the same pair (j, k) and switches blocks B'_i, D'_i with B_i, D_i. This proves that the map is an involution. $\square$

We now consider three other classes of polynomials, which we denote by $e_n^{[\leq 2]}(x)$, $e_n^{[\geq 2]}(x)$ and $e_n^{[=2]}(x)$. Their definition is similar to that of the exponential polynomials $e_n(x)$, but we now restrict the sum to partitions with block sizes less than or equal to two, greater than or equal to two, and equal to two, respectively. Some properties of these polynomials are:

(i) $\sum_{n \geq 0} e_n^{[\leq 2]}(x) t^n / n! = \exp(x(t + t^2/2))$,

(ii) $\sum_{n \geq 0} e_n^{[\geq 2]}(x) t^n / n! = \exp(x(e^t - 1 - t))$,

(iii) $\sum_{n \geq 0} e_n^{[=2]}(x) t^n / n! = \exp(x t^2/2)$,

(iv) $e_n^{[\leq 2]}(1)$ is the number of involutions on an n-element set,

(v) $e_n^{[\geq 2]}(1)$ is the number of partitions on an n-element set without singleton blocks, and

(vi) $e_n^{[=2]}(x) = (n-1) \cdot (n-3) \cdots 1 \cdot x^{n/2}$ if n is even, otherwise equal to zero.

By examing the proof of Theorem 1, we see that it also applies to partitions with block sizes less than or equal to two, greater than or equal to two, and equal to two. Hence we have the surprising result:

Theorem 3 *The Hankel determinants of the polynomials $e_n^{[\leq 2]}(x)$, $e_n^{[\geq 2]}(x)$, and $e_n^{[=2]}(x)$ are given by*

$$\det\left(e_{i+j}^{[\leq 2]}(x)\right)_{0\leq i,j\leq n} = \det\left(e_{i+j}^{[\geq 2]}(x)\right)_{0\leq i,j\leq n} = \det\left(e_{i+j}^{[=2]}(x)\right)_{0\leq i,j\leq n} = x^{(n+1)n/2} \cdot \prod_{i=0}^{n} i!.$$

The first Hankel determinant in Theorem 3 was evaluated by Radoux in [4].

By considering permutations π on the set S that are 'compatible' with permutations σ, one can give a bijective proof of the following theorem. We leave the details to the reader. The first Hankel determinant is classical, the second is due to Radoux [3].

Theorem 4 *Let $D(n)$ denote the number of derangements of an n-element set, that is, the number of permutations without fixed points. The Hankel determinants of order $n+1$ of the matrices $[(i+j)!]$ and $[D(i+j)]$ are given by*

$$\det\left((i+j)!\right)_{0\leq i,j\leq n} = \det\left(D(i+j)\right)_{0\leq i,j\leq n} = \left(\prod_{i=0}^{n} i!\right)^2.$$

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References

[1] M. Gardner, *Fractal Music, Hypercards and more: Mathematical Recreations from Scientific American,* W. H. Freeman and Company, New York, 1992.

[2] C. Radoux, Calcul effectif de certains déterminants de Hankel, *Bull. Soc. Math. Belg. Sér. B* **31** (1979) 49–55.

[3] C. Radoux, Déterminant de Hankel construit sur des polynomes liés aux nombres de dérangements, *European J. Combin.* **12** (1991) 327–329.

[4] C. Radoux, Déterminants de Hankel et théorème de Sylvester, in *Séminaire Lotharingien de Combinatoire (Saint-Nabor, 1992), Publ. Inst. Rech. Math. Av., 498, Univ. Louis Pasteur, Strasbourg, 1992,* pp. 115–122.

[5] C. Radoux, Addition formulas for polynomials built on classical combinatorial sequences, to appear in *J. Comput. Appl. Math.*

[6] G.-C. Rota, *Finite Operator Calculus,* Academic Press, Inc., New York, 1975.

[7] G.-C. Rota, D. Kahaner, and A. Odlyzko, On the foundations of combinatorial theory. VIII. Finite operator calculus, *J. Math. Anal. Appl.* **42** (1973) 684–760.

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