The Möbius function of partitions with restricted block sizes[*]

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Abstract

The purpose of this paper is to compute the Möbius function of filters in the partition lattice formed by restricting to partitions by type. The Möbius function is determined in terms of the descent set statistics on permutations and the Möbius function of filters in the lattice of integer compositions. When the underlying integer partition is a knapsack partition, the Möbius function on integer compositions is determined by a topological argument. In this proof the permutahedron makes a cameo appearance.

MSC: 05A17; 05A18; 06A07

Keywords: Euler and tangent numbers; descent set statistic; set partition lattice; r-divisible partition lattice; knapsack partitions; permutahedron

1   Introduction

Sylvester [9] initiated the study of subposets of the partition lattice. He proved that the Möbius function of the poset of set partitions where each block has even cardinality is given by every other Euler number, also known as the tangent numbers. Recall the Euler numbers enumerate alternating permutations. Stanley [6] extended this result by considering set partitions where each block has cardinality divisible by r. In this case the Möbius function is given by the number of permutations with descent set $\{r, 2r, 3r, \ldots\}$. In this paper we continue to explore the connection between partition lattices and permutation statistics.

To each set partition we can assign a type, which is the integer partition consisting of the multiset of cardinalities of the blocks. Given a set F of integer partitions it is then natural to ask for the Möbius function of the associated poset of set partitions whose types belong to F. We slightly modify this by instead working with pointed set partitions and pointed integer partitions and letting F be a filter in the poset of pointed integer partitions.

In Theorem 3.1 we consider pointed set partitions whose types belong to a given filter F of pointed integer partitions. The question of computing the Möbius function is reduced to the descent
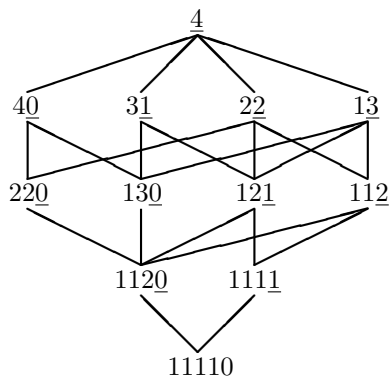

Figure 1: The poset I_4 of pointed partitions of the integer 4.

set statistics and Möbius functions in the smaller and more tractable lattice of integer compositions. In Section 4 we consider knapsack partitions, a notion motivated by a well-known cryptosystem. Theorem 4.4 allows us to determine the Möbius function of the integer composition lattice. We obtain explicit expressions for the sought-after Möbius function in Theorem 4.5.

2   Pointed integer partitions, set partitions and compositions

Recall that an integer partition $\lambda = \{\lambda_1, \ldots, \lambda_k\}$ of a non-negative integer n is a multiset of positive integers having sum n, that is, $n = \lambda_1 + \cdots + \lambda_k$.

Definition 2.1 *Let n be a non-negative integer. A pointed integer partition of n is a pair $\{\lambda, \underline{m}\} = \{\lambda_1, \ldots, \lambda_k, \underline{m}\}$ where m is a non-negative integer and $\lambda = \{\lambda_1, \ldots, \lambda_k\}$ is an integer partition of $n - m$. The integer m is called the pointed part. It is underlined to distinguish it from the other parts of the partition.*

Denote by $I_n^\bullet$ the collection of all pointed integer partitions of the non-negative integer n. Partially order the set $I_n^\bullet$ by the two cover relations

$$\{\lambda_1, \ldots, \lambda_{k-2}, \lambda_{k-1}, \lambda_k, \underline{m}\} \prec \{\lambda_1, \ldots, \lambda_{k-2}, \lambda_{k-1} + \lambda_k, \underline{m}\}$$

and

$$\{\lambda_1, \ldots, \lambda_{k-1}, \lambda_k, \underline{m}\} \prec \{\lambda_1, \ldots, \lambda_{k-1}, \underline{\lambda_k + m}\}.$$

In other words, by adding two parts together we go up in the order. If one of the parts is the pointed part then the sum becomes the pointed part in the new pointed partition. Observe that the poset $I_n^\bullet$ is not a lattice for $n \geq 3$.

Definition 2.2 *A pointed set partition $\pi = (\sigma, Z)$ of a finite set S consists of a subset Z of S and a partition σ of the set difference $S - Z$.*

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We call the subset Z the *zero block* of the pointed set partition π. Moreover, we denote the number of blocks (including the zero block) of π by $|\pi|$. Let $\Pi_n^\bullet$ be the poset of all pointed set partitions on the set $\{1, \ldots, n\}$, where the partial order is given by refinement. That is, for two pointed set partitions π and π', we have that $\pi \leq \pi'$ if every block of π is contained in some block (possibly the zero block) of π' and the zero block of π is contained in the zero block of π'. Observe that $\Pi_n^\bullet$ is isomorphic to the partition lattice Π_{n+1} by inserting the element $n+1$ into the zero block and letting the zero block be an ordinary block of the partition.

We define the *type* of a pointed partition (σ, Z) to be the pointed integer partition $\{\lambda, \underline{m}\}$ where m is the cardinality of the zero block Z and λ is the multiset of the sizes of the blocks of σ, that is, $\lambda = \{|B| \; : \; B \in \sigma\}$.

Definition 2.3 *Let n be a non-negative integer. A pointed integer composition of n is a list $\vec{c} = (c_1, \ldots, c_{k-1}, \underline{c_k})$ of non-negative integers with sum n where c_1 through c_{k-1} are required to be positive.*

Note that the last entry c_k is allowed to be 0, so we underline it to distinguish it from the other entries. Let $C_n^\bullet$ be the collection of all pointed compositions of n. Partially order the elements of $C_n^\bullet$ by the cover relations

$$(c_1, \ldots, c_{j-1}, c_j, c_{j+1}, c_{j+2}, \ldots, c_{k-1}, \underline{c_k}) \prec (c_1, \ldots, c_{j-1}, c_j + c_{j+1}, c_{j+2}, \ldots, c_{k-1}, \underline{c_k}),$$

$$(c_1, \ldots, c_{k-2}, c_{k-1}, \underline{c_k}) \prec (c_1, \ldots, c_{k-2}, \underline{c_{k-1} + c_k}).$$

That is, the cover relation occurs by adding two adjacent entries of the composition. Observe that the poset $C_n^\bullet$ is isomorphic to the Boolean algebra on n elements.

For a permutation τ in the symmetric group on n elements $\mathfrak{S}_n$, the *descent set* is a subset of $\{1, \ldots, n-1\}$ defined as $\{i \; : \; \tau(i) > \tau(i+1)\}$. Given a permutation τ in $\mathfrak{S}_n$ with descent set $\{s_1, s_2, \ldots, s_{k-1}\}$ where $s_1 < s_2 < \cdots < s_{k-1}$, define the descent composition of τ to be $(s_1, s_2 - s_1, \ldots, s_{k-1} - s_{k-2}, n - s_{k-1})$. For a composition $\vec{c} = (c_1, \ldots, c_k)$ of n let $\beta(\vec{c})$ denote the number of permutations in the symmetric group $\mathfrak{S}_n$ with descent composition $\vec{c}$, that is, with the descent set $\{c_1, c_1 + c_2, \ldots, c_1 + \cdots + c_{k-1}\}$. For a pointed composition $\vec{c} = (c_1, \ldots, c_{k-1}, \underline{c_k})$ with c_k strictly greater than 0, let $\beta(\vec{c})$ be as in the non-pointed composition case. If the pointed composition $\vec{c} = (c_1, \ldots, c_{k-1}, \underline{c_k})$ satisfies $k \geq 2$ and $c_k = 0$, let $\beta(\vec{c}) = 0$. Lastly, for $\vec{c} = (\underline{0})$ let $\beta(\vec{c}) = 1$.

A uniform way to view the descent composition statistic independently of the last part is the following. For a pointed composition $\vec{c} = (c_1, \ldots, c_{k-1}, \underline{c_k})$ let $\vec{d}$ be the composition where the last part is incremented by one, that is, $\vec{d} = (c_1, \ldots, c_{k-1}, c_k + 1)$. Then $\beta(\vec{c})$ is the number of permutations τ in $\mathfrak{S}_{n+1}$ having descent composition $\vec{d}$ and satisfying $\tau(n+1) = n+1$.

The type of an integer composition $(c_1, \ldots, c_{k-1}, \underline{c_k})$ is the pointed integer partition $\{c_1, \ldots, c_{k-1}, \underline{c_k}\}$. Observe that the last part of the composition becomes the pointed part.

We remark that the three posets $I_n^\bullet$, $\Pi_n^\bullet$, and $C_n^\bullet$ are graded, and hence ranked. Throughout we will denote the rank function of a graded poset by ρ and use $\rho(x, y)$ to denote the rank difference $\rho(x, y) = \rho(y) - \rho(x)$.

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3   The Möbius function of restricted partitions

Recall that a *filter* F (also known as an upper order ideal) in a poset Q is a subset of Q such that if $x \leq y$ and x belongs to F then y belongs to F. For S a subset of the poset Q, the filter generated by S is given by $\{y \in Q : \exists\, x \in S \text{ such that } x \leq y\}$. Note that if f is an order-preserving map from a poset P to a poset Q and F is a filter of Q then the inverse image $f^{-1}(F)$ is a filter of P. For further information about posets, we refer the reader to Stanley's treatise [7].

Let F be a filter in the pointed integer partition poset $I_n^\bullet$. Let $\Pi_n^\bullet(F)$ be the filter of the pointed set partition lattice $\Pi_n^\bullet$ consisting of all set partitions having their types belonging to F, that is,

$$\Pi_n^\bullet(F) = \{\pi \in \Pi_n^\bullet \ : \ \text{type}(\pi) \in F\}.$$

Similarly, define $C_n^\bullet(F)$ to be the filter of pointed compositions having types belonging to F, that is,

$$C_n^\bullet(F) = \{\vec{c} \in C_n^\bullet \ : \ \text{type}(\vec{c}) \in F\}.$$

Observe that both $\Pi_n^\bullet(F)$ and $C_n^\bullet(F)$ are join semi-lattices. Hence after adjoining a minimal element $\hat{0}$ we have that both $\Pi_n^\bullet(F) \cup \{\hat{0}\}$ and $C_n^\bullet(F) \cup \{\hat{0}\}$ are lattices.

Theorem 3.1 *Let F be a filter of the pointed integer partition poset $I_n^\bullet$. Then the Möbius function of the filter $\Pi_n^\bullet(F)$ with a minimal element $\hat{0}$ adjoined is given by*

$$\mu\left(\Pi_n^\bullet(F) \cup \{\hat{0}\}\right) = \sum_{\vec{c} \in C_n^\bullet(F)} (-1)^{\rho(\vec{c},\hat{1})} \cdot \mu_{C_n^\bullet(F) \cup \{\hat{0}\}}(\hat{0}, \vec{c}) \cdot \beta(\vec{c}). \tag{3.1}$$

Although $C_n^\bullet(F) \cup \{\hat{0}\}$ is not necessarily graded, the expression $\rho(\vec{c}, \hat{1})$ appearing in the statement of Theorem 3.1 is well-defined since the interval $[\vec{c}, \hat{1}]$ is itself graded.

Proof of Theorem 3.1: For a pointed composition $\vec{c} = (c_1, \ldots, c_{k-1}, \underline{c_k})$ of a non-negative integer n recall that the multinomial coefficient $\binom{n}{\vec{c}}$ is defined as $n!/(c_1! \cdots c_k!)$. Observe that $\binom{n}{\vec{c}}$ counts the number of permutations in $\mathfrak{S}_n$ having descent set contained in the set $\{c_1, c_1 + c_2, \ldots, c_1 + \cdots + c_{k-1}\}$. By the principle of inclusion and exclusion, we have that $\beta(\vec{c}) = \sum_{\vec{c} \leq \vec{d}} (-1)^{\rho(\vec{c}, \vec{d})} \cdot \binom{n}{\vec{d}}$. Thus the right-hand side of (3.1) is given by

$$\sum_{\vec{c} \in C_n^\bullet(F)} (-1)^{\rho(\vec{c}, \hat{1})} \cdot \mu_{C_n^\bullet(F) \cup \{\hat{0}\}}(\hat{0}, \vec{c}) \cdot \beta(\vec{c})$$

$$= \sum_{\vec{c} \in C_n^\bullet(F)} \sum_{\vec{c} \leq \vec{d}} (-1)^{\rho(\vec{d}, \hat{1})} \cdot \mu_{C_n^\bullet(F) \cup \{\hat{0}\}}(\hat{0}, \vec{c}) \cdot \binom{n}{\vec{d}}$$

$$= \sum_{\vec{d} \in C_n^\bullet(F)} (-1)^{\rho(\vec{d}, \hat{1})} \cdot \binom{n}{\vec{d}} \cdot \sum_{\hat{0} < \vec{c} \leq \vec{d}} \mu_{C_n^\bullet(F) \cup \{\hat{0}\}}(\hat{0}, \vec{c})$$

$$= - \sum_{\vec{d} \in C_n^\bullet(F)} (-1)^{\rho(\vec{d}, \hat{1})} \cdot \binom{n}{\vec{d}}.$$

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The last sum can be viewed as follows. An *ordered set partition* is a partition in $\Pi_n^\bullet$ with an ordering of the blocks such that the last block is the zero block. The type of an ordered set partition is the pointed composition where one lists the size of each block in their given order. Note that given a composition $\vec{d}$ of n, there are $\binom{n}{\vec{d}}$ ordered set partitions with $\vec{d}$ as its type. Hence we have that

$$
\begin{aligned}
- \sum_{\vec{d} \in C_n^\bullet(F)} (-1)^{\rho(\vec{d}, \hat{1})} \cdot \binom{n}{\vec{d}} &= \sum_{\substack{\pi \text{ ordered set partition} \\ \text{type}(\pi) \in C_n^\bullet(F)}} (-1)^{|\pi|} \\
&= \sum_{\pi \in \Pi_n^\bullet(F)} (-1)^{|\pi|} \cdot (|\pi| - 1)! \\
&= - \sum_{\pi \in \Pi_n^\bullet(F)} \mu_{\Pi_n^\bullet(F) \cup \{\hat{0}\}}(\pi, \hat{1}) \\
&= \mu(\Pi_n^\bullet(F) \cup \{\hat{0}\}). \qquad \square
\end{aligned}
$$

Example 3.2 We have the following identity connecting the Stirling numbers of the second kind with the Eulerian numbers:

$$
- \sum_{j=1}^{k} (-1)^{j-1} \cdot (j-1)! \cdot S(n+1, j) = (-1)^k \cdot \sum_{j=1}^{k} \binom{n-j}{n-k} \cdot A(n, j). \tag{3.2}
$$

Recall that the Stirling number of the second kind $S(n, j)$ counts the number of set partitions of an n-element set into j parts, whereas the Eulerian number $A(n, j)$ counts the number of permutations in $\mathfrak{S}_n$ with $j - 1$ descents. To prove equation (3.2) let F be the filter of $I_n^\bullet$ consisting of all pointed integer partitions with at most k parts. Then the filter $\Pi_n^\bullet(F)$ consists of all pointed set partitions with at most k parts. Using the Stirling numbers of the second kind, we can write the Möbius function of $\Pi_n^\bullet(F) \cup \{\hat{0}\}$ to be the left-hand side of equation (3.2). Similarly, the filter $C_n^\bullet(F)$ consists of all pointed compositions of n with at most k parts. Let $\vec{c}$ be a composition with j parts. Then the interval $[\hat{0}, \vec{c}]$ in the poset $C_n^\bullet(F) \cup \{\hat{0}\}$ is a rank-selected Boolean algebra, more specifically, the Boolean algebra B_{n-j+1} with ranks 1 through $n - k$ removed. Hence the Möbius function $\mu_{C_n^\bullet(F) \cup \{\hat{0}\}}(\hat{0}, \vec{c})$ is given by $(-1)^{k-j+1} \cdot \binom{n-j}{n-k}$. Summing over all pointed compositions $\vec{c}$ consisting of j parts in the right-hand side in Theorem 3.1, we obtain $(-1)^k \cdot \binom{n-j}{n-k}$ times the number of permutations in $\mathfrak{S}_n$ with $j - 1$ descents. Hence the identity follows. When $k = n$ this identity states that $\mu(\Pi_{n+1}) = (-1)^n \cdot n!$ since $\Pi_n^\bullet(F) \cup \{\hat{0}\} \cong \Pi_n^\bullet \cong \Pi_{n+1}$.

We have the following corollary. This result was proved with different techniques in [5].

Corollary 3.3 *Let* $n = r \cdot p + m$. *Let* $\Pi_n^{\bullet, r, m}$ *be all the partitions in* $\Pi_n^\bullet$ *where the zero block has cardinality at least* m *and the remaining blocks have cardinality divisible by* r. *Then the Möbius function of* $\Pi_n^{\bullet, r, m} \cup \{\hat{0}\}$ *is given by*

$$
\mu\left(\Pi_n^{\bullet, r, m} \cup \{\hat{0}\}\right) = (-1)^{p+1} \cdot \beta(\underbrace{r, r, \ldots, r}_{p}, \underline{m}).
$$

Proof: Let F be the filter of $I_n^\bullet$ generated by the pointed partition $\{r, r, \ldots, r, \underline{m}\}$. The poset $\Pi_n^{\bullet, r, m}$ is exactly the filter $\Pi_n^\bullet(F)$. Observe that $C_n^\bullet(F) \cup \{\hat{0}\}$ has a unique atom, namely the pointed composition $(r, r, \ldots, r, \underline{m})$. Thus the Möbius function $\mu_{C_n^\bullet(F) \cup \{\hat{0}\}}(\hat{0}, \vec{c})$ is non-zero only for this composition. Hence the summation in the right-hand side of (3.1) has only one term, namely $(-1)^{p+1} \cdot \beta(r, r, \ldots, r, \underline{m})$. $\square$

By setting $m = r - 1$ in the previous corollary and using the bijection between $\Pi_n^\bullet$ and Π_{n+1}, we obtain the following corollary due to Stanley [6, 8].

Corollary 3.4 (Stanley) *Let $n = r \cdot p$ and let Π_n^r denoted the r-divisible lattice, that is, all partitions on n elements where the cardinality of each block is divisible by r and a minimal element $\hat{0}$ is adjoined. Then the Möbius function of Π_n^r, $\mu(\Pi_n^r)$, is given by the sign $(-1)^p$ times the number of permutations τ in $\mathfrak{S}_n$ with descent set $\{r, 2r, \ldots, n - r\}$ and $\tau(n) = n$.*

4 Knapsack partitions

Let $\lambda = \{e_1^{m_1}, e_2^{m_2}, \ldots, e_q^{m_q}\}$ be a partition, that is, a multiset of positive integers, where m_i denotes the multiplicity of the element e_i. We tacitly assume that all the e_i's are distinct, that is, $e_i \neq e_j$ for $i \neq j$. Since there are $\prod_{i=1}^q (m_i + 1)$ multi-subsets μ of λ, the following inequality holds:

$$\left| \left\{ \sum_{e \in \mu} e \ : \ \mu \subseteq \lambda \right\} \right| \leq \prod_{i=1}^q (m_i + 1). \tag{4.1}$$

When equality holds in (4.1), we call λ *a knapsack partition.* Observe this is equivalent to that each integer in the set on the left-hand side of (4.1) has a unique representation as a sum of elements from the multiset λ. When all the entries in λ are distinct, that is, λ is a set, this definition reduces to the usual notion of a knapsack system appearing in cryptography.

Example 4.1 *(a) If $e_1, \ldots, e_q, m_1, \ldots, m_q$ are positive integers satisfying the inequality $\sum_{i=1}^{j-1} m_i \cdot e_i \leq e_j$ for all $j = 2, \ldots, q$ then $\{e_1^{m_1}, \ldots, e_q^{m_q}\}$ is a knapsack partition.*

(b) If $\{\lambda_1, \ldots, \lambda_p\}$ is a knapsack partition, q is a prime greater than the sum $\lambda_1 + \cdots + \lambda_p$ and j is a positive integer less than q, then $\{j \cdot \lambda_1 \bmod q, \ldots, j \cdot \lambda_p \bmod q\}$ is a knapsack partition.

A pointed integer partition $\{\lambda, \underline{m}\}$ is called a *pointed knapsack partition* if λ is a knapsack partition.

In this section we consider the filter F generated by a pointed knapsack partition $\{\lambda, \underline{m}\}$. We determine the Möbius function of the poset $C_n^\bullet(F) \cup \{\hat{0}\}$ and thus obtain an explicit formula for $\mu(\Pi_n^\bullet(F) \cup \{\hat{0}\})$. Before proceeding one more definition is needed. Let $V(\lambda, \underline{m}) = V$ be the collection of all pointed compositions $\vec{c} = (c_1, \ldots, c_{k-1}, \underline{m})$ in the filter $C_n^\bullet(F)$ such that when each c_i, $1 \leq i \leq k-1$, is expressed as a sum of parts of λ, the summands for each c_i are distinct.

Example 4.2 For the pointed knapsack partition $\{1, 1, 1, 4, \underline{m}\}$ the set V of pointed compositions is

$$\begin{aligned} V \ = \ & \{(1, 1, 1, 4, \underline{m}), (1, 1, 5, \underline{m}), (1, 1, 4, 1, \underline{m}), (1, 5, 1, \underline{m}), \\ & (1, 4, 1, 1, \underline{m}), (5, 1, 1, \underline{m}), (4, 1, 1, 1, \underline{m})\}. \end{aligned}$$

Observe the composition $(2, 1, 4, \underline{m})$ does not belong to V since 2 is the sum of two equal parts.

Example 4.3 For the pointed knapsack partition $\{r, r, \ldots, r, \underline{m}\}$ the set V only consists of the pointed composition $(r, r, \ldots, r, \underline{m})$.

The ordered partition lattice Q_p consists of all ordered partitions of the set $\{1, \ldots, p\}$ together with a minimal element $\hat{0}$ adjoined. The cover relation in Q_p is to merge two adjacent blocks. The ordered partition lattice is isomorphic to the face lattice of the $(p-1)$-dimensional permutahedron; see for instance [1] or Exercise 2.9 in [2]. Hence Q_p is Eulerian and has Möbius function given by $\mu_{Q_p}(x, y) = (-1)^{\rho(x,y)}$.

Theorem 4.4 *Let F be the filter in the pointed integer partition poset $I_n^\bullet$ generated by the pointed knapsack partition $\{\lambda, \underline{m}\} = \{\lambda_1, \ldots, \lambda_p, \underline{m}\}$ of the integer n. Let $\vec{c} = (c_1, \ldots, c_{k-1}, \underline{c_k})$ be a pointed composition in the filter $C_n^\bullet(F)$. Then the Möbius function $\mu_{C_n^\bullet(F) \cup \{\hat{0}\}}(\hat{0}, \vec{c})$ is given by*

$$\mu(\hat{0}, \vec{c}) = \begin{cases} (-1)^{p-k} & \text{if } \vec{c} \in V, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Consider the lattice $C_n^\bullet(F) \cup \{\hat{0}\}$ and let $\vec{c}$ be a pointed composition in $C_n^\bullet(F)$. Assume that the pointed part of the composition $\vec{c}$ is greater than m. The interval $[\hat{0}, \vec{c}]$ in $C_n^\bullet(F) \cup \{\hat{0}\}$ is itself a lattice and each atom in this interval has pointed part equal to m. The join of these atoms also has pointed part equal to m, so the join cannot be equal to the composition $\vec{c}$. By Corollary 3.9.5 in [7], the Möbius function vanishes, that is, $\mu(\hat{0}, \vec{c}) = 0$.

Let P be the ideal of $C_n^\bullet(F)$ consisting of all pointed compositions with pointed part m. Observe that P has the maximal element $(\lambda_1 + \cdots + \lambda_p, \underline{m})$. Hence P is a finite join semi-lattice and thus $P \cup \{\hat{0}\}$ is a lattice.

We first consider the case when all the parts of λ are distinct. In this case $P = V$. There is an isomorphism between the lattice $P \cup \{\hat{0}\}$ and the ordered partition lattice Q_p. The isomorphism f sends the ordered partition $(B_1, B_2, \ldots, B_k)$ to the pointed composition

$$f((B_1, B_2, \ldots, B_k)) = \left(\sum_{i \in B_1} \lambda_i, \sum_{i \in B_2} \lambda_i, \ldots, \sum_{i \in B_k} \lambda_i, \underline{m} \right),$$

and $f(\hat{0}) = \hat{0}$. Hence we conclude that the Möbius function is given by $\mu(\hat{0}, \vec{c}) = (-1)^{p-k}$.

For the general case we allow the partition to contain multiple entries. The ideal P is then isomorphic to a filter of the ordered partition lattice Q_p. Namely, let R be the collection of all ordered partitions $(B_1, B_2, \ldots, B_k)$ such that if $\lambda_i = \lambda_j$ for $1 \le i < j \le p$ then the elements i and j either appear in the same block or the element i appears in a block before the block containing the element j. It is straightforward to verify that R is a filter of Q_p and that the map f is again an isomorphism, this time from R to P.
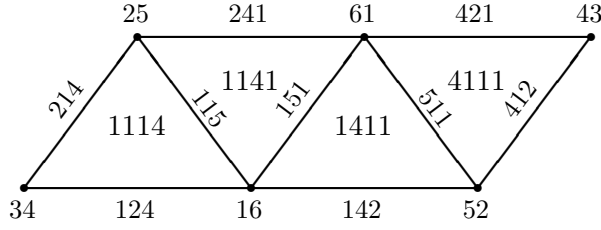
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Figure 2: The complex Γ corresponding to the pointed knapsack partition $\{1,1,1,4,\underline{m}\}$. In this figure the pointed part $\underline{m}$ has been omitted. Observe the faces not on the boundary of Γ correspond to the set V. Refer to Example 4.2.

Recall that the boundary of the dual of the permutahedron is a simplicial complex Δ_p. We view this simplicial complex as the $(p-2)$-dimensional sphere in the hyperplane $K = \{(x_1,\dots,x_p) \in \mathbb{R}^p : x_1 + \cdots + x_p = 0\}$ cut by the $\binom{p}{2}$ hyperplanes $x_i = x_j$ for $1 \le i < j \le p$. The geometric picture is the sphere S^{p-2} with $\binom{p}{2}$ great spheres on it. Let $\mathcal{H}$ be the hyperplane arrangement in $\mathbb{R}^p$ given by the collection of the hyperplanes $x_i = x_j$ when $\lambda_i = \lambda_j$. Let $\mathcal{C}$ be the chamber

$$\mathcal{C} = \{(x_1,\dots,x_p) \in K \ : \ x_i \le x_j \text{ if } i < j \text{ and } \lambda_i = \lambda_j\}.$$

Note that $\mathcal{C}$ is a cone in K.

The filter R in the poset Q_p corresponds to the subcomplex Γ of the complex Δ_p consisting of all faces G in Δ_p that are contained in the chamber $\mathcal{C}$. The geometric realization of Γ is the intersection of the unit sphere S^{p-2} and the chamber $\mathcal{C}$, and hence is homeomorphic to a $(p-2)$-dimensional ball. Let $\mathcal{L}(\Gamma)$ denote the face lattice of Γ.

A face G is on the boundary of Γ if and only if G is contained in one of the hyperplanes in $\mathcal{H}$. In other words, G is on the boundary of Γ if and only if the corresponding pointed composition $\vec{c} = (c_1,\dots,c_{k-1},\underline{m})$ has an entry c_i, $i < k$, such that when c_i is expressed uniquely as a sum of parts of λ, two terms are equal. In this case the pointed composition $\vec{c}$ does not belong to the set V.

For a composition $\vec{c}$ let G be the associated face in Γ. Then we have that

$$\mu(\hat{0}, \vec{c}) = \mu_{\mathcal{L}(\Gamma)}(G, \hat{1}) = \begin{cases} 0 & \text{if } G \text{ is on the boundary of } \Gamma, \\ \widetilde{\chi}(\Gamma) = 0 & \text{if } G \text{ is the empty face}, \\ (-1)^{\rho(G,\hat{1})} & \text{otherwise}, \end{cases}$$

where the last step is Proposition 3.8.9 in [7], proving the theorem. $\square$

Combining Theorems 3.1 and 4.4, we have:

Theorem 4.5 *Let F be the filter in the pointed integer partition poset $I_n^{\bullet}$ generated by the pointed knapsack partition $\{\lambda,\underline{m}\} = \{\lambda_1,\dots,\lambda_p,\underline{m}\}$ of the integer n. Then the Möbius function $\mu(\Pi_n^{\bullet}(F) \cup \{\hat{0}\})$ of the filter $\Pi_n^{\bullet}(F)$ with a minimal element $\hat{0}$ adjoined is given by*

$$\mu(\Pi_n^{\bullet}(F) \cup \{\hat{0}\}) = (-1)^{p-1} \cdot \sum_{\vec{c} \in V} \beta(\vec{c}),$$

that is, $(-1)^{p-1}$ times the number of permutations in $\mathfrak{S}_n$ whose descent composition belongs to the set V.

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Observe that Corollary 3.3 is also a consequence of Theorem 4.5 using Example 4.3. In this case the complex Γ is a simplex.

Continuation of Example 4.2 *Let F be the filter in $I_n^\bullet$ generated by the pointed knapsack partition $\{1,1,1,4,\underline{m}\}$. Then*

$$
\begin{aligned}
-\mu\left(\Pi_n^\bullet(F)\cup\{\hat{0}\}\right) \;=\; &\beta(1,1,1,4,\underline{m}) + \beta(1,1,5,\underline{m}) + \beta(1,1,4,1,\underline{m}) \\
&+\beta(1,5,1,\underline{m}) + \beta(1,4,1,1,\underline{m}) + \beta(5,1,1,\underline{m}) \\
&+\beta(4,1,1,1,\underline{m}).
\end{aligned}
$$

5   Concluding remarks

The poset $\Pi_n^\bullet(F)\cup\{\hat{0}\}$ raises many natural questions. When the minimal elements of the filter F have the same rank, the poset $\Pi_n^\bullet(F)\cup\{\hat{0}\}$ is graded. One may ask if this is a shellable poset. Similarly, for a general filter, that is, when the minimal elements of the filter F have different ranks, the previous question extends to determining if the order complex of $\Pi_n^\bullet(F)\cup\{\hat{0}\}$ is non-pure shellable [3]. In the case when the poset is not shellable, can one still determine the homology groups of the order complex? One goal here is to obtain a bijective proof of Theorem 4.5.

The symmetric group $\mathfrak{S}_n$ acts on the order complex of $\Pi_n^\bullet(F)\cup\{\hat{0}\}$. This action is inherited by the homology groups. Can this representation be determined? When one considers a pointed knapsack partition it is natural to conjecture that the poset is indeed shellable and hence the homology is concentrated in the top homology. Furthermore, the dimension of the top homology is given by the Möbius function and the action of the symmetric group is given by the direct sum of Specht modules corresponding to the skewpartitions associated with the compositions in the set V. For the pointed knapsack partition $\{r,r,\ldots,r,\underline{r-1}\}$ corresponding to the r-divisible lattice (recall Corollary 3.4), this research program has been carried out. See the papers [4] and [10] and the references therein.

Finally, an enumerative question is to determine the number of knapsack partitions of n. This number seems related to the prime factor decomposition of n. Techniques from analytic number theory may be required.

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