Exam 1

Name: ______________________ Section and/or TA: ______________

Last Four Digits of Student ID: __________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used except for a one-page sheet of formulas and facts. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. Unsupported answers on free response problems will receive *no credit*.

Multiple Choice Questions

1 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ **6** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

2 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ **7** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

3 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ **8** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

4 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ **9** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

5 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ **10** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

SCORE

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

Multiple Choice Questions

1. $\int x \ln(x)\, dx =$

   A. $\dfrac{x^2}{2} \ln x - \frac{1}{4}x^2 + C$

   B. $\ln x + 1 + C$

   C. $x^3 \ln x + C$

   D. $x^2 \ln x + x^2 + C$

   E. $x^2 \ln x + C$

2. Which of the following is the correct form of the partial fraction expansion of

$$\frac{x^5 + 1}{x(x-1)(x^2+1)^2}\,?$$

   A. $\dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{Cx+D}{(x^2+1)^2}$

   B. $\dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{(Cx+D)^2}{(x^2+1)^2}$

   C. $\dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{Cx+D}{x^2+1)}$

   D. $\dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{Cx+D}{(x^2+1)} + \dfrac{(Ex+F)^2}{(x^2+1)^2}$

   E. $\dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{Cx+D}{x^2+1} + \dfrac{Ex+F}{(x^2+1)^2}$

3. Use the fact that

$$\frac{x^2 - x + 6}{x^3 + 3x} = -\frac{x+1}{x^2+3} + \frac{2}{x}$$

to evaluate the integral

$$\int \frac{x^2 - x + 6}{x^3 + 3x}\, dx.$$

A. $\log(x^2+3) - \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \log(x^2) + C$

B. $\frac{1}{\sqrt{3}}\log(x^2+3) - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \log(x^2) + C$

C. $-\frac{1}{2}\log(x^2+3) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \log(x^2) + C$

D. $\frac{1}{\sqrt{3}}\log(x^2+3) - \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + +\log(x^2)C$

E. $\log(x^2+3) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \log(x^2) + C$

4. Find $\int_0^{\pi/2} \sin^2 x \cos^3 x\, dx$.

A. 0

B. $-2/15$

C. $\pi/4$

D. $2/15$

E. $-\pi/4$

5. Find $\int_0^1 \frac{dx}{(x^2+1)^2}$.

   A. $-\pi/8$

   B. $1/4 - \pi/8$

   C. $\pi/8$

   D. $1/4 + \pi/8$

   E. $-1/4 + \pi/8$

6. Find the value of

$$\int_0^1 x^2 f''(x)\,dx$$

   if

$$\int_0^1 f(x)\,dx = 0$$

   and

$$f(1) = 3, \quad f'(1) = 0, \quad f''(0) = f''(1) = 0$$

   A. -6

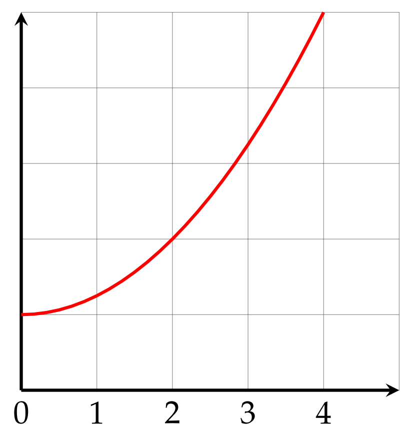
   B. -3

   C. 6

   D. 3

   E. 0

7. The graph of the function f is shown below for $0 \leq x \leq 4$. Suppose that we approximate $\int_0^4 f(x)\,dx$ with $N = 4$ intervals using the left, right, and midpoint methods. Which of the following choices correctly describes the relationship between L_4, R_4, and M_4?



A. $L_4 \leq R_4 \leq M_4$

B. $L_4 \leq M_4 \leq R_4$

C. $R_4 \leq L_4 \leq M_4$

D. $R_4 \leq M_4 \leq L_4$

E. $M_4 \leq R_4 \leq L_4$

8. Use Simpson's rule with $N = 4$ to approximate $\int_0^2 f(x)\,dx$ if f has the values shown in the table below.

x	0	0.5	1	1.5	2.0
$f(x)$	12	6	36	6	12

A. 6

B. 72

C. 144

D. 12

E. 24

9. What are all the values of p for which $\displaystyle\int_0^1 \frac{1}{x^{2p}}\,dx$ converges?

A. $p > -1$

B. $p > 0$

C. $p < \frac{1}{2}$

D. $p < 1$

E. There are no values of p for which this integral converges.

10. A certain sequence $\{a_n\}$ is defined recursively by the formula

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{3}{a_n}\right)$$

and converges to a limit $A > 0$. Find the value of A.

A. $\sqrt{2}$

B. $\sqrt{3}$

C. 2

D. 3

E. 1

Free Response Questions

11. Determine whether each of the following sequences is convergent or divergent. If convergent, find the limit.

(a) (3 points) $\dfrac{2n^2+1}{5n^2+n}$

Solution:

$$\lim_{n\to\infty} \frac{2n^2+1}{5n^2+n} = \lim_{n\to\infty} \frac{n^2\left(2+\frac{1}{n^2}\right)}{n^2\left(5+\frac{1}{n}\right)}$$
$$= \frac{2}{5}$$

(converges)

(b) (4 points) $\ln(n+1) - \ln(n)$

Solution:

$$\lim_{n\to\infty} \ln(n+1) - \ln(n) = \lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right)$$
$$= \ln\left(\lim_{n\to\infty} \frac{n+1}{n}\right)$$
$$= \ln(1) = 0$$

(converges)

(c) (3 points) $\dfrac{n^3+n+1}{n+2}$

Solution:

$$\lim_{n\to\infty} \frac{n^3+n+1}{n+2} = \lim_{n\to\infty} \frac{n^3\left(1+\frac{1}{n^2}+\frac{1}{n^3}\right)}{n\left(1+\frac{2}{n}\right)}$$
$$= \infty$$

(diverges)

12. Find the following antiderivatives:

(a) (4 points) $\int x \cos 5x \, dx$

Solution: Integrate by parts with $u = x$, $dv = \cos 5x \, dx$, so $v = \frac{1}{5}\sin 5x$, $du = dx$. Then

$$\begin{aligned}\int x \cos 5x \, dx &= \frac{x}{5}\sin 5x - \int \frac{1}{5}\sin 5x \, dx \\ &= \frac{x}{5}\sin 5x + \frac{1}{25}\cos 5x + C\end{aligned}$$

(b) (6 points) $\int (x^2+1)e^{-x} \, dx$

Solution: We'll need two integrations by parts. Let $u = x^2+1$, $dv = e^{-x}$, so $v = -e^{-x}$, $du = 2xdx$. Then

$$\int (x^2+1)e^{-x} \, dx = -(x^2+1)e^{-x} + \int 2xe^{-x} \, dx + C$$

To compute the second term, let $u = 2x$, $dv = e^{-x}$ so that $du = 2\,dx$ and $v = -e^{-x}$. Then

$$\begin{aligned}\int 2xe^{-x} \, dx &= -2xe^{-x} + \int 2e^{-x} \, dx \\ &= -2xe^{-x} - 2e^{-x} + C\end{aligned}$$

Putting this together we get

$$\int (x^2+1)e^{-x} \, dx = -(x^2+1)e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

13. Find the following integrals

(a) (5 points) $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$

Solution: First let $u = \ln x$, so that $du = \dfrac{1}{x}\,dx$. Integration by substitution gives

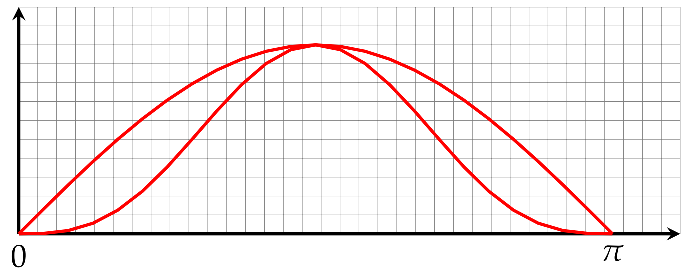
$$\begin{aligned}
\lim_{t\to\infty}\int_e^t \frac{1}{x(\ln x)^2}\,dx &= \lim_{t\to\infty}\int_1^{\ln t}\frac{1}{u^2}\,du \\
&= \lim_{t\to\infty}\left[-\frac{1}{u}\right]_{u=1}^{u=\ln t} \\
&= \lim_{t\to\infty}\left(1-\frac{1}{\ln t}\right) \\
&= 1
\end{aligned}$$

(b) (5 points) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Solution:

$$\begin{aligned}
\lim_{t\to 1}\int_0^t \frac{dx}{\sqrt{1-x^2}} &= \lim_{t\to 1}(\arcsin(t)-0) \\
&= \frac{\pi}{2}
\end{aligned}$$

14. Find the area between the curves $y = \sin x$ and $y = \sin^3 x$ for $0 \leq x \leq \pi$.



Solution: The curve $y = \sin x$ is above the curve $y = \sin^3 x$ so the area is

$$\begin{aligned} A &= \int_0^{\pi} \left[\sin x - \sin^3 x\right] dx \\ &= \int_0^{\pi} \sin x \left(1 - \sin^2 x\right) dx \\ &= \int_0^{\pi} \cos^2 x \sin x \, dx \end{aligned}$$

If we substitute $u = \cos x$, $du = -\sin x \, dx$ we get

$$\begin{aligned} A &= -\int_1^{-1} u^2 \, du \\ &= \int_{-1}^{1} u^2 \, du \\ &= 2\int_0^1 u^2 \, du \\ &= \frac{2}{3} \end{aligned}$$

15. Recall that the error bound for the Trapezoid rule states that, if you compute $\int_a^b f(x)\,dx$ using the Trapezoid rule with n subintervals, the difference between the true and approximate value of the integral is at most

$$\frac{K(b-a)^3}{12n^2}$$

where K is an upper bound for $|f''(x)|$ on $[a, b]$. We'll consider

$$\int_1^2 \frac{1}{x}\,dx$$

(a) (4 points) Find $f''(x)$ if $f(x) = 1/x$ and find the maximum value of $|f''(x)|$ on $[1, 2]$.

Solution: By two differentiations we get $f''(x) = -2/x^3$, so $|f''(x)| = 2/x^3$. This function is decreasing so its maximum value occurs at $x = 1$ and is 2.

(b) (6 points) How large should we take n to calculate $\int_1^2 1/x\,dx$ using the Trapezoid rule to two decimal place accuracy (i.e., with error no more than 0.005)? For K, use the maximum value of $|f''(x)]$ found in part (a).

Solution: Taking $K = 2$, $a = 1$, $b = 2$, we see that

$$E_T \le \frac{2 \cdot 1^3}{12n^2} = \frac{1}{6n^2}.$$

We want to choose n so that

$$\frac{1}{6n^2} \le 0.005$$

or

$$6n^2 \ge 200$$

Trial and error shows that $n = 5$ falls short at $6n^2 = 150$ but $n = 6$ passes the test with $6n^2 = 216$. Hence $n = 6$.