Exam 2

Name: ____________________________ Section and/or TA: ________________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. The wise student will show work for the multiple choice problems as well.

Multiple Choice Questions

1	A	B	C	D	E	**6**	A	B	C	D	E
2	A	B	C	D	E	**7**	A	B	C	D	E
3	A	B	C	D	E	**8**	A	B	C	D	E
4	A	B	C	D	E	**9**	A	B	C	D	E
5	A	B	C	D	E	**10**	A	B	C	D	E

SCORE

Multiple Choice	11	12	13	14	15	Total Score
50	9	8	5	16	12	100

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Multiple Choice Questions

1. A series $\sum a_n$ is convergent if and only if

   A. the limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ is greater than 1.

   B. its sequence of terms $\{a_n\}$ converges to 0.

   C. its sequence of partial sums $\{s_n\}$ converges to some real number.

   D. its sequence of terms $\{a_n\}$ is alternating.

   E. its sequence of partial sums $\{s_n\}$ is bounded.

2. The power series expanded around $a = 0$ for the function $f(x) = \frac{3}{2+4x}$ is

   A. $\sum_{n=0}^{\infty} (-1)^n 3 \cdot 2^{n-1} x^n$.

   B. $\sum_{n=0}^{\infty} 3 \cdot 2^{n-1} x^n$.

   C. $3 \sum_{n=0}^{\infty} (-4)^n x^n$.

   D. $\frac{3}{4} \sum_{n=0}^{\infty} (-1)^n x^n$.

   E. $\frac{3}{4} \sum_{n=0}^{\infty} x^n$.

3. Consider the following statements.

   I. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent.

   II. By the alternating series test the series $\sum_{n=1}^{\infty} (-1)^n n^3$ is convergent.

   III. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is conditionally convergent.

   A. Only I is true.

   B. Only II is true.

   C. Only III is true.

   D. Only I and III are true.

   E. Only II and III are true.

4. What is the value of $\sum_{n=1}^{\infty} \frac{2^n}{3^{n-1}}$?

   A. 1

   B. 2

   C. 4

   D. 6

   E. The series diverges.

5. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{14}{n^2 + 2n}$$

is convergent or divergent by expressing S_n as a telescoping sum. If convergent, find its sum.

A. diverges

B. 2

C. $\frac{1}{14}$

D. $\frac{2}{21}$

E. $\frac{21}{2}$

6. Give the interval of convergence for the series

$$\sum_{n=1}^{\infty} \frac{(x-4)^{2n}}{n^2}.$$

A. $(3,5)$

B. $[3,5)$

C. $[3,5]$

D. $[3,5)$

E. None of the above.

7. The sequence $\{a_n\}$ is given recursively by $a_{n+1} = \dfrac{a_n}{1+a_n}$ and $a_1 = 2$. The first five terms of this sequence are:

   A. $2, \dfrac{2}{3}, \dfrac{2}{5}, \dfrac{2}{7}, \dfrac{2}{9}$

   B. $2, \dfrac{2}{3}, \dfrac{2}{12}, \dfrac{2}{60}, \dfrac{2}{360}$

   C. $2, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \dfrac{5}{6}$

   D. $2, \dfrac{2}{3}, \dfrac{2}{5}, \dfrac{2}{12}, \dfrac{2}{14}$

   E. $2, \dfrac{2}{3}, \dfrac{2}{5}, \dfrac{2}{15}, \dfrac{2}{75}$

8. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n\sin^2 n}{1+n^3}$. Applying the comparison test with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ leads to the following conclusion.

   A. The test is inconclusive.

   B. The series converges absolutely.

   C. The series converges conditionally.

   D. The series diverges.

   E. The test cannot be applied to $a_n = \dfrac{n\sin^2 n}{1+n^3}$ and $b_n = \dfrac{1}{n^2}$.

9. The series $\sum_{n=1}^{\infty} \frac{1}{n^n}$

   A. diverges by the Ratio Test.

   B. diverges by the Integral Test.

   C. converges by the Limit Comparison Test with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

   D. diverges by the Limit Comparison Test with the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

   E. converges by the Root Test.

10. Which one of the following series converges for all real numbers x?

   A. $\sum_{n=1}^{\infty} \frac{x^n}{n}$

   B. $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

   C. $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

   D. $\sum_{n=1}^{\infty} \frac{e^n x^n}{n!}$

   E. $\sum_{n=1}^{\infty} \frac{n! x^n}{e^n}$

Free Response Questions

11. Determine if the sequence is convergent or divergent. If convergent give its limit.

(a) (3 points) $a_n = \dfrac{3+5n^2}{n+n^2}$

Solution: The sequence converges

$$\lim_{n\to\infty} \frac{3+5n^2}{n+n^2} = \lim_{n\to\infty} \frac{\frac{3}{n^2}+5}{\frac{1}{n}+1} = 5.$$

(b) (3 points) $a_n = e^{2n/(n+2)}$

Solution: The sequence converges.

$$\lim_{n\to\infty} e^{2n/(n+2)} = e^2.$$

(c) (3 points) $a_n = \dfrac{\arctan(n)}{n}$

Solution: The sequence converges.

$$\lim_{n\to\infty} \frac{\arctan(n)}{n} = \frac{\pi/2}{\lim_{n\to\infty} n} = 0.$$

12. (a) (4 points) State the ratio test. Be sure your conclusion describes all three cases.

Solution: Given a series $\sum_{n=1}^{\infty} a_n$ let

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L.$$

1. If $L < 1$ the series is absolutely convergent (and therefore convergent).
2. If $L > 1$ the series is divergent.
3. If $L = 1$ the test is inconclusive.

(b) (4 points) Use the ratio test to determine if the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges. Explain your reasoning.

Solution:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n} = \lim_{n\to\infty}\frac{2}{n+1} = 0 < 1$$

Since $L < 1$ the series converges.

13. (5 points) Use the integral test to determine whether the series

$$\sum_{n=1}^{\infty} n^2 e^{-n^3}$$

converges or diverges. *Show your work.*

Solution: Use the Integral Test with $f(x) = x^2 e^{-x^3}$. Let $u = x^3$ and $du = 3x^2\,dx$

$$\begin{aligned}
\int_1^{\infty} x^2 e^{-x^3} dx &= \lim_{M\to\infty} \int_1^{M} x^2 e^{-x^3} dx \\
&= \lim_{M\to\infty} \frac{1}{3} \int_1^{M} e^{-u}\,du \\
&= \lim_{M\to\infty} -\frac{1}{3} e^{-u}\Big|_1^M = \frac{1}{3e} < \infty
\end{aligned}$$

Since the integral converges, the series converges.

14. (a) (4 points) Determine the power series expansion about $a = 0$ for the function $f(x) = \dfrac{1}{1+x^4}$.

Solution: Since $\dfrac{1}{1-x} = \sum x^n$, replacing x by $-x^4$, we have

$$\frac{1}{1+x^4} = \sum_{n=0}^{\infty} (-x^4)^n = \sum_{n=0}^{\infty} (-1)^n x^{4n}.$$

(b) (4 points) Determine the power series expansion about $a = 0$ for $\displaystyle\int_0^x \frac{1}{1+t^4} dt$.

Solution: We find the power series expansion for this by integrating the power series from (a) term-by-term

$$\int_0^x \frac{1}{1+t^4} dt = \sum_{n=0}^{\infty} \int_0^x (-1)^n t^{4n} dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{4n+1} = x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \cdots$$

(c) (4 points) Use the first three nonzero terms to estimate $\displaystyle\int_0^{1/2} \frac{1}{1+t^4} dt$.

Solution:

$$\int_0^{1/2} \frac{1}{1+t^4} dt \approx \frac{1}{2} + \frac{1}{5 \cdot 2^5} - \frac{1}{9 \cdot 2^9} = \frac{11659}{23040} = 0.50603298\overline{611}$$

(d) (4 points) Using the Alternating Series Estimation Theorem, estimate the error in using the sum in part (c).

Solution: Since this is an alternating series, the error is bounded by the next term in the series which is

$$\frac{1}{13 \cdot 2^{13}} = \frac{1}{106496} = 0.0000093900240\overline{384615}.$$

15. A function f is defined by

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

for all x in the interval of convergence for the power series.

(a) (4 points) Find the radius of convergence for the power series. ***Show your work.***

Solution: Use the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2n+1}{2n+3} \frac{|x^{2n+3}|}{|x^{2n+1}|} = |x^2| < 1.$$

To converge we must have $x^2 < 1$, so the radius of convergence is 1.

(b) (4 points) Find the interval of convergence for the power series. ***Show your work.***

Solution: The radius of convergence is 1, so we need to check the endpoints: $x = 1$ and $x = -1$. At $x = -1$, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^{3n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

which converges by the Alternating Series Test. At $x = 1$ we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

which converges by the Alternating Series Test.
Thus, the interval of convergence is $-1 \le x \le 1$ or $[-1, 1]$.

(c) (4 points) Find the power series representation for $f'(x)$ and state its radius of convergence.

Solution:

$$f'(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots .$$

The radius of convergence does not change and remains at 1.