EXAM 2

Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. You are allowed to use notes on a single piece of 8.5”X11” paper, front and back, including formulas and theorems. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1 (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E)

6 (A) (B) (C) (D) (E)

7 (A) (B) (C) (D) (E)

8 (A) (B) (C) (D) (E)

9 (A) (B) (C) (D) (E)

10 (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

Multiple Choice Questions

1. (5 points) Give the first five terms of the sequence $\{a_1, a_2, \ldots\}$ defined by

$$a_n = \frac{\cos(n\pi)}{n^2}.$$

A. $\{\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}\}$
B. $\{\frac{-1}{1}, \frac{1}{8}, \frac{-1}{27}, \frac{1}{64}, \frac{-1}{125}\}$
C. $\{\frac{-1}{1}, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}\}$
D. $\{\frac{-1}{1}, \frac{1}{4}, \frac{-1}{9}, \frac{1}{16}, \frac{-1}{25}\}$
E. $\{\frac{1}{1}, \frac{-1}{4}, \frac{1}{9}, \frac{-1}{16}, \frac{1}{25}\}$

2. (5 points) Find the limit of the sequence $\{a_1, a_2, \ldots\}$ defined by

$$a_n = \frac{n+1}{\sqrt{n^2+5}}.$$

A. 1
B. 2
C. 0
D. $\frac{1}{5}$
E. $\frac{1}{2}$

3. (5 points) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converge or diverge?

A. Diverges by a comparison test to $\sum_{n=1}^{\infty} \frac{1}{2^n}$.
B. Converges because it is a geometric series and $|r| < 1$.
C. Converges by the limit comparison test to $\sum_{n=1}^{\infty} \frac{1}{2^n}$.
D. Converges because $\displaystyle\lim_{n\to\infty} \frac{1}{2^n - 1} = 0$.
E. Diverges by a comparison test to $\sum_{n=1}^{\infty} \frac{1}{n}$.

4. (5 points) Which of the following series converge?

A. $\sum_{n=1}^{\infty} \frac{(-1)^n}{2}$

B. $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$

C. $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}$

D. $\sum_{n=2}^{\infty} \frac{1}{\ln(n^{\frac{3}{2}})}$

E. None of the above series converge.

5. (5 points) Find the sum of the series $\sum_{n=1}^{\infty} \left(\frac{1}{2^n} + \frac{1}{4^n} \right)$

A. $\frac{10}{3}$

B. $\frac{3}{4}$

C. $\frac{7}{3}$

D. $\frac{4}{3}$

E. This series is divergent.

6. (5 points) What would you compare $\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{n^3+1}$ to for a conclusive limit comparison test?

A. $\sum_{n=1}^{\infty} \ln n$

B. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

C. $\sum_{n=1}^{\infty} \frac{1}{n}$

D. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

E. The limit comparison test can't be used to understand convergence for this series.

7. (5 points) Does the series $\sum_{n=1}^{\infty} \frac{n^2}{(2n)!}$ converge or diverge?

A. Converges by the ratio test because $\lim_{n\to\infty} \frac{(n+1)^2}{n^2(2n+2)(2n+1)} = 0$

B. Converges by the ratio test because $\lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 1$

C. Converges by the ratio test because $\lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = 1$

D. Diverges by the ratio test because $\lim_{n\to\infty} \frac{n^2(2n+2)(2n+1)}{(n+1)^2} > 1$

E. Diverges by the ratio test because $\lim_{n\to\infty} \frac{n^2(2n+2)(2n+1)}{(n+1)^2} > 0$

8. (5 points) What is the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2}$?

   A. $[0, 2)$
   B. $[-1, 1)$
   C. $[0, 2]$
   D. $[-1, 1]$
   E. $\{1\}$

9. (5 points) Which power series represents **An Antiderivative** of $\frac{1}{1-x^3}$ on the interval $(-1, 1)$?

   A. $\sum_{n=0}^{\infty} \frac{x^{3n+1}}{3n+1}$
   B. $\sum_{n=0}^{\infty} 3n x^{3n-1}$
   C. $\sum_{n=0}^{\infty} (-1)^n x^{3n}$
   D. $\sum_{n=0}^{\infty} x^{3n}$
   E. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n}$

10. (5 points) Find the first 3 nonzero terms of the Taylor series for $f(x) = \cos(\pi x)$ centered at 0.

   A. $-\frac{\pi}{1!}x + \frac{\pi^3}{3!}x^3 - \frac{\pi^5}{5!}x^5$
   B. $1 + \frac{\pi}{1!}x + \frac{\pi^2}{2!}x^2$
   C. $1 - \frac{\pi}{1!}x + \frac{\pi^2}{2!}x^2$
   D. $\frac{\pi}{2!}x^2 + \frac{\pi^3}{4!}x^3 + \frac{\pi^5}{6!}x^5$
   E. $1 - \frac{\pi^2}{(2!)}x^2 + \frac{\pi^4}{4!}x^4$

Free Response Questions

11. (a) (5 points) Use $\dfrac{2}{4n^2-1} = \dfrac{1}{2n-1} - \dfrac{1}{2n+1}$ to find a simpler expression for the $k-th$ partial sum

$$\sum_{n=1}^{k} \frac{1}{4n^2-1}.$$

Solution: This is a telescoping series. $\frac{1}{2}\sum_{n=1}^{k} \dfrac{1}{2n-1} - \dfrac{1}{2n+1} = \dfrac{1}{2}\left(1 - \dfrac{1}{4k+1}\right)$

(b) (5 points) What is the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1}?$$

Solution: The sum is the limit of the partial sums: $\lim_{k\to\infty} \dfrac{1}{2}\left(1 - \dfrac{1}{4k+1}\right) = \dfrac{1}{2}$

12. Are the series below absolutely convergent, conditionally convergent, or divergent? Justify your answer.

(a) (4 points) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5n^3}{3+n+7n^3}\right)^n$

Solution: Use the root test: $\displaystyle\lim_{n\to\infty}\left|\left(\frac{5n^3}{3+n+7n^3}\right)^n\right|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{5n^3}{3+n+7n^3} = \frac{5}{7} < 1$, so this series absolutely converges.

(b) (6 points) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2-2n}{n^3+1}$

Solution: This series conditionally converges by the alternating series test:
$\displaystyle\lim_{n\to\infty}\frac{n^2-2n}{n^3+1} = 0$
$\displaystyle\left(\frac{x^2-2x}{x^3+1}\right)' =$ is negative when $x \geq 10$,

But the absolute value series diverges by a limit comparison test with $\sum_{n=1}^{\infty}\frac{1}{n}$. So the series is conditionally convergent.

13. (a) (5 points) What is the **radius** of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{4^n} x^n$?

Solution: The ratio test shows that this series converges absolutely when

$$\lim_{n\to\infty} \frac{(n+1)^2+1}{4^{n+1}} \frac{4^n}{n^2+1} |x| = \frac{1}{4}|x| < 1$$

So the radius of convergence is 4.

(b) (5 points) What is the **radius** of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^2}(x-1)^n$? What is the **interval** of convergence? Clearly label your answers.

Solution: The ratio test shows that this series converges absolutely when

$$\lim_{n\to\infty} \frac{(n+1)!}{(n+1)^2} \frac{n^2}{n!} |x-1| = \lim_{n\to\infty} \frac{n^2}{n+1} |x-1| < 1$$

This happens only when $x = 1$. So the radius is 0 and the interval is $[1, 1]$.

14. (a) (6 points) Find the first 2 nonzero terms in the Taylor series for $\sqrt{1+x^2}$ centered at zero.

Solution: We take some derivatives and plug in 0:
$\sqrt{1+x^2}$

$$\frac{d}{dx}\left(\sqrt{1+x^2}\right)' = \frac{x}{\sqrt{1+x^2}}$$
$$\frac{d}{dx}\left(\sqrt{1+x^2}\right)'' = \frac{1}{(1+x^2)^{\frac{3}{2}}}$$

so $f(0) = 1$, $f'(0) = 0$, $f''(0) = 1$, and the Taylor series starts $1 + \frac{1}{2}x^2 + ...$

(b) (4 points) Use your solution to part (a) to estimate $\int_0^1 \sqrt{1+x^2}dx$.

Solution: Take the integral $\int_0^1 1 + \frac{1}{2}x^2\ dx$ to get $[x + \frac{1}{6}x^3]_0^1 = 1 + \frac{1}{6}$.

15. (a) (5 points) Find the Taylor series for the function $f(x) = \dfrac{x}{1-x^2}$ centered at 0.

Solution: This is $x\left(\dfrac{1}{1-x^2}\right) = x\sum_{n=0}^{\infty}(x^2)^n = \sum_{n=0}^{\infty} x^{2n+1}$

(b) (5 points) Use your answer in part (a) to find the Taylor series centered at 0 for the function $g(x) = \ln(1-x^2)$. (**Hint:** It will help to find the antiderivative of $f(x)$ in part (a).)

Solution: $-\dfrac{1}{2}\ln(1-x^2) = \displaystyle\int \frac{x}{1-x^2}\,dx = \sum_{n=0}^{\infty}\int x^{2n+1}dx = \sum_{n=0}^{\infty}\frac{1}{2n+2}x^{2n+2}$