# *EXAM 3*

Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. You are allowed to use notes on a single piece of 8.5" x 11" paper, front and back, including formulas and theorems. **You are required to turn this page in with your exam.** You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS). Absolutely no communication device use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show <u>all work</u> to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

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Multiple Choice Questions

**1** (A) (B) (C) (D) (E) **6** (A) (B) (C) (D) (E)

**2** (A) (B) (C) (D) (E) **7** (A) (B) (C) (D) (E)

**3** (A) (B) (C) (D) (E) **8** (A) (B) (C) (D) (E)

**4** (A) (B) (C) (D) (E) **9** (A) (B) (C) (D) (E)

**5** (A) (B) (C) (D) (E) **10** (A) (B) (C) (D) (E)

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| Multiple Choice | 11 | 12 | 13 | 14 | 15 | Total Score |
|---|---|---|---|---|---|---|
| 50 | 10 | 10 | 10 | 10 | 10 | 100 |
| | | | | | | |

1. (5 points) What is the average value of the function $f(x) = (\sin(x))^2 \cos(x)$ on the interval $[0, \frac{\pi}{2}]$?

   A. $\dfrac{2\pi}{3}$

   **B.** $\dfrac{2}{3\pi}$

   C. $-\dfrac{\pi}{3}$

   D. 0

   E. $\dfrac{3\pi}{2}$

2. (5 points) Recall that the circle of radius 2 centered at the point $(2, 3)$ is the set of points $(x, y)$ which satisfy:
$$(x-2)^2 + (y-3)^2 = 4.$$
   Which of the following is a parametrization of the circle of radius 2 centered at $(2, 3)$?

   A. $x(t) = 2 - \cos(3t)$, $y(t) = 3 - \sin(3t)$

   **B.** $x(t) = 2 + 2\sin(5t)$, $y(t) = 3 + 2\cos(5t)$

   C. $x(t) = 2 + 2\sin(t)$, $y(t) = 3 - 3\cos(t)$

   D. $x(t) = 1 + 2\cos(t)$, $y(t) = 1 + 3\sin(t)$

   E. $x(t) = 2\sqrt{1-t}$, $y(t) = 3\sqrt{1+t}$

3. (5 points) Three masses are located in the plane: 1 gram at $(0, 2)$, 1 gram at $(-10, 2)$, and 3 grams at $(1, -5)$. Find the center of mass of this system.

   **A.** $(-\frac{7}{5}, -\frac{11}{5})$

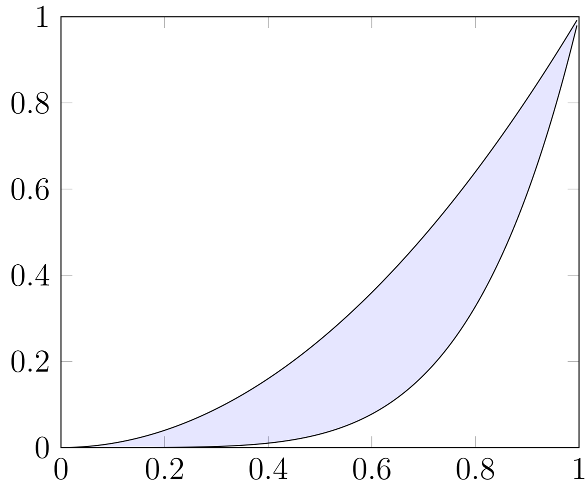
   B. $(-1, -2)$

   C. $(0, -\frac{12}{5})$

   D. $(0, -\frac{9}{5})$

   E. $(\frac{6}{5}, -\frac{14}{5})$

4. (5 points) The region bounded by $y = x^2$ and $y = x^5$ is shown below.




Consider the solid obtained by rotating this region around the **x-axis**. Using the **disks/washers** method, which integral will compute the volume of this solid?

**A.** $\int_0^1 \pi \left(x^4 - x^{10}\right) dx$**.**

B. $\int_0^{\sqrt{3}} 2\pi \left(x^2 - x^5\right)^2 dx$.

C. $\int_0^1 \pi \left(x^2 - x^5\right) dx$.

D. $\int_0^1 \pi \left(x^2 - x^5\right)^2 dx$.

E. $\int_0^1 2\pi \left(x^3 - x^6\right) dx$.

5. (5 points) Which integral below computes the length of the curve $y = f(x)$ where $f(x) = \cos(x) + \sin(x)$, and $0 \leq x \leq \pi$?

A. $\int_0^\pi \sqrt{1 + \cos^2(x) - \sin^2(x)}\, dx$.

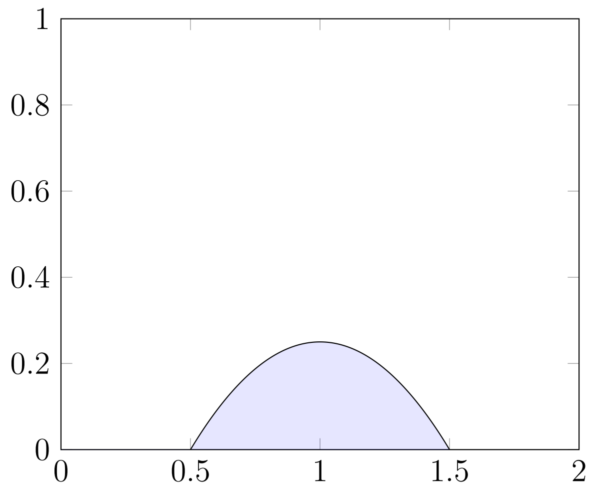
B. $\int_0^\pi \sqrt{t^2 + [\cos(x) - \sin(x)]^2}\, dx$.

**C.** $\int_0^\pi \sqrt{2 - 2\sin(x)\cos(x)}\, dx$**.**

D. $\int_0^\pi \sqrt{1 + [2\cos(x))]^2}\, dx$.

E. $\int_0^\pi 2\pi t^2 \sqrt{1 + [\cos(x) + \sin(x)]^2}\, dx$.

6. (5 points) The region bounded by the curve $y = -x^2 + 2x - \frac{3}{4}$ and the $x$-axis is shown below.




Consider the solid obtained by rotating this region about the **y-axis**. Using the **shell** method, which integral will compute the volume of this solid?

A. $\pi \int_{\frac{1}{2}}^{\frac{3}{2}} (-x^3 + 2x^2 - \frac{3}{4}x)^2 \, dx.$

**B.** $2\pi \int_{\frac{1}{2}}^{\frac{3}{2}} (-x^3 + 2x^2 - \frac{3}{4}x) \, dx.$

C. $\pi \int_{\frac{1}{2}}^{\frac{3}{2}} (-x^3 + 2x^2 - \frac{3}{4}x) \, dx.$

D. $2\pi \int_{\frac{1}{2}}^{\frac{3}{2}} (-x^6 + 2x^4 - \frac{3}{4}x^2) \, dx.$

E. $2\pi \int_{\frac{1}{2}}^{\frac{3}{2}} (-x^2 + 2x - \frac{3}{4}) \, dx.$

7. (5 points) Find the slope of the tangent line to the curve parametrized by $x(t) = t^2, \quad y(t) = t^3 - 2t$ at the point $(x, y) = (9, 21)$.

A. $-5$

B. $\frac{27}{5}$

**C.** $\frac{25}{6}$

D. $\frac{27}{2}$

E. $-\frac{6}{25}$

8. (5 points) The line $y = 2x$ for $1 \leqslant x \leqslant 2$ is rotated about the **x-axis**. What is the **surface area** of the resulting surface?

   A. $30\pi$

   B. $18\sqrt{3}\pi$

   **C.** $6\sqrt{5}\pi$

   D. $20\pi$

   E. $\sqrt{31}\pi$

9. (5 points) Let $R$ be the region bounded by $f(x) = e^x$, $y = e$ and the $y$-axis; and form a solid by revolving R about the **y-axis**. Which integral represents the volume of this solid using the **shell method**?

   **A.** $2\pi \int_0^1 x(e - e^x)\, dx$

   B. $2\pi \int_0^1 e^{2x}\, dx$

   C. $2\pi \int_0^1 \ln(x)\, dx$

   D. $2\pi \int_1^e x\sqrt{1 + e^{2x}}\, dx$

   E. $2\pi \int_0^1 xe^{2x}\, dx$

10. (5 points) The curve $y = \sqrt{x}$ for $0 \leqslant x \leqslant 1$ is rotated about the **y-axis**, producing a surface. Which of the following integrals calculates its surface area?

   A. $\int_0^1 2\pi\sqrt{1 + \frac{1}{4y}}\, dy$

   B. $\int_{-1}^0 2\pi y^2\sqrt{1 + \left(\frac{1}{y}\right)^2}\, dy$

   C. $\int_0^1 2\pi y^2\sqrt{1 + y^2}\, dy$

   D. $\int_0^1 2\pi\sqrt{y + \frac{1}{4}}\, dy$

   **E.** $\int_0^1 2\pi y^2\sqrt{1 + 4y^2}\, dy$

Free Response Questions

11. The *cycloid* for the circle of radius 1 is the curve parametrized by the functions

$$x(\theta) = \theta - \sin(\theta),$$
$$y(\theta) = 1 - \cos(\theta).$$

(a) (2 points) Find the coordinates of the point $P(\theta) = (x(\theta), y(\theta))$ when $\theta = \frac{\pi}{4}$.

**Solution:** We note that $\sin(\pi/4) = \sqrt{2}/2$ so that

$$x(\pi/4) = \pi/4 - \sqrt{2}/2, \quad y(\pi/4) = 1 - \sqrt{2}/2.$$

So we get

$$P(\pi/4) = (\pi/4 - \sqrt{2}/2, 1 - \sqrt{2}/2).$$

1 point for each coordinate.

(b) (4 points) Find the **slope** of the tangent line to the cycloid at the point $P(\frac{\pi}{4})$ from part $(a)$.

**Solution:**

$\dfrac{dx}{d\theta} = x'(\theta) = 1 - \cos(\theta), \qquad \dfrac{dy}{d\theta} = y'(\theta) = \sin(\theta).$

$x'(\dfrac{\pi}{4}) = 1 - \dfrac{1}{\sqrt{2}}, \qquad y'(\dfrac{\pi}{4}) = \sin(\dfrac{\pi}{4}) = \dfrac{1}{\sqrt{2}}.$

$\dfrac{dy}{dx} = \dfrac{y'(\frac{\pi}{4})}{x'(\frac{\pi}{4})} = \dfrac{\frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} = \dfrac{1}{\sqrt{2} - 1}$. 1 point for each derivative, 1 point for each numerical evaluation of $x'$ and $y'$.

(c) (4 points) Set up <u>but do not evaluate</u> the integral to find the arc length of the piece of the cycloid parametrized by $0 \leq \theta \leq \pi$ (you may assume that the curve is traced only once).

**Solution:** $L = \displaystyle\int_0^\pi \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta = \int_0^\pi \sqrt{(1 - \cos(\theta))^2 + (\sin(\theta))^2} d\theta$

After expanding the argument of the square root, one gets:

$$L = \sqrt{2} \int_0^\pi \sqrt{1 - \cos\theta} d\theta.$$

2 points for the general form of the integral, 2 points for the correct evaluation of the integrand on the right. It is not necessary to obtain the last formula.

12. Let $S$ be the region in the plane bounded by $y = 4 - x^2$ and the $x$-axis for $-2 \leq x \leq 2$. Assume that $S$ has uniform density $\rho = 1$.

(a) (8 points) Find the total mass $M$ and the moments $M_y$ and $M_x$ for $S$.
**Clearly label** each of your answers.

**Solution:** The total mass is

$$m = \rho \int_{-2}^{2} \sqrt{4 - x^2} dx = 2\rho\pi = 2\pi,$$

since the area is one-half the area of a disk of radius $r = 2$. The moments are

$$M_y = 2\rho \int_{0}^{2} x\sqrt{4 - x^2}\ dx = 16/3,$$

$$M_x = \frac{\rho}{2} \int_{-2}^{2} (4 - x^2)\ dx = \int_{0}^{2} (4 - x^2) dx = 16/3.$$

2 points for the mass, 3 points for each moment: 2 points for the integral and 1 point for the results for each moment.

(b) (2 points) Find the center of mass of $S$.

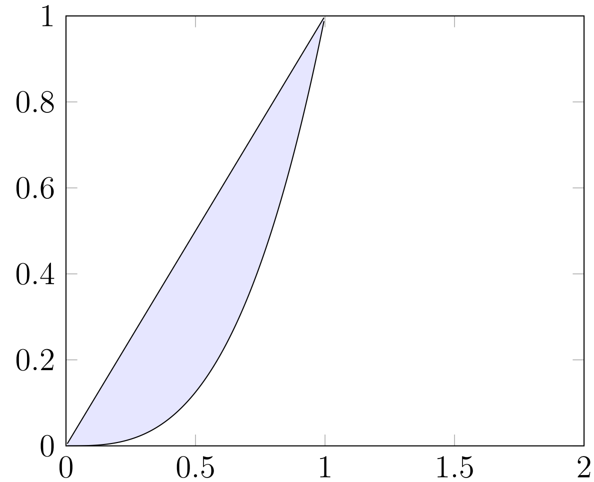
**Solution:**

$$\overline{x} = \frac{M_y}{m} = \frac{1}{2}$$

$$\overline{y} = \frac{M_x}{m} = \frac{1}{2}.$$

1 point for each coordinate of the center of mass (centroid).

13. The region between the curves $y = x$ and $y = x^3$ is shown below.




Let $V$ be the solid of revolution obtained by rotating this region **around the y-axis**.

(a) (5 points) Write an integral which computes the volume of $V$ using the disk/washer method, and then evaluate the integral.

**Solution:** Slice the solid perpendicular to the $y$-axis. The cross-sectional area id the difference of the area of two disks:

$$A(y) = \pi(y^{2/3} - y^2)\ dy$$

so that the volume is

$$V = \int_{y=0}^{1} A(y)\ dy = \pi \int_0^1 (y^{2/3} - y^2)\ dy = \frac{4}{15}\pi.$$

3 points for the cross-sectional area, and 2 points for the volume integral and its evaluation.

(b) (5 points) Write an integral which computes the volume of $V$ using the cylindrical shells method, and then evaluate the integral.

**Solution:** For the shell method, integrate with respect to $x$-axis. The elementary shell volume is

$$V(x) = 2\pi x(x - x^3)\Delta x,$$

so that the volume is

$$V = \int_{x=0}^{1} V(x)\ dx = 2\pi \int_0^1 x(x - x^3)\ dx = \frac{4}{15}\pi.$$

3 points for the cross-sectional area, and 2 points for the volume integral and its evaluation.

14. Let $L$ be the arc parametrized by $x(t) = t^2$, $y(t) = t^3$, $0 \leq t \leq 1$.

(a) (6 points) Find the arc length of $L$.

**Solution:**

$$x'(t) = 2t, \qquad y'(t) = 3t^2.$$

The arc length is

$$L = \int_0^1 \sqrt{4t^2 + 9t^4}\, dt = \int_0^1 2t\sqrt{1 + (9/4)t^2}\, dt,$$

Let $u = 1 + (9/4)t^2$ so $du = (9/2)t\, dt$, then the integral is

$$L = \frac{4}{9}\int_1^{13/4} \sqrt{u}\, du = \frac{8}{27}\left[\left(\frac{13}{4}\right)^{3/2} - 1\right].$$

2 points for the derivatives, 2 points for the correct integral, and 2 points for the integration and solution.

(b) (4 points) Set up but do not evaluate an integral which computes the area of the surface $S_1$ obtained by revolving $L$ around the **x-axis**.

**Solution:**
The elementary arc length along the curve is:

$$ds = 2t\sqrt{1 + (9/4)t^2}\, dt,$$

so the surface area is

$$S = \int_0^1 \pi x^2(t) ds(t) = 2\pi \int_0^1 t^5\sqrt{1 + (9/4)t^2}\, dt.$$

1 point for the infinitesimal arc length expression, 1 point for the first integral expression, and 1 point for the final result.

15. Let V be the solid whose base is the circle centered at the origin of radius 1, with cross sections given by squares perpendicular to the **x-axis**.

(a) (3 points) Find a function giving the area of the cross-section of $V$ at $x$.

**Solution:** The length of a side of the square is $2\sqrt{1-x^2}$ as $x$ varies in $[0,1]$. The cross-sectional area is then $A(x) = (2\sqrt{1-x^2})^2 = 4(1-x^2)$. 2 points for the side length, 1 point for the area.

(b) (4 points) Set up an integral which computes the volume of $V$.

(c) (3 points) Find the volume of $V$.

**Solution:** Taking $x \in [0,1]$ will sweep out one-half of the volume so

$$V = 2\int_{x=0}^{1} A(x)\ dx = 8\int_0^1 (1-x^2)\ dx = \frac{16}{3}.$$

3 points for the correct expression for $V$, 2 points for integrating, and 1 point for the correct result.