Problem 1.
5. **(5 points)** `local/rmb-problems/e3/arc-length-num.pg`

Find the length of the curve $y = \frac{2}{3}x^{3/2}$ between $x = 8$ and $x = 24$.
The length is ______

Exact answers are preferred. Your answer must be correctly rounded to three decimal places, or more accurate.
Solution: *(Instructor solution preview: show the student solution after due date.)*

SOLUTION
If $y = \frac{2}{3}x^{3/2}$, then $y' = \sqrt{x}$. The length is given by the integral $\int_8^{24} \sqrt{1+x}\,dx$. The anti-derivative of $(1+x)^{1/2}$ is $(2/3)(1+x)^{3/2}$. Thus the length of the curve will be

$$\int_8^{24} \sqrt{1+x}\,dx = \frac{2}{3}(1+x)^{3/2}\Big|_8^{24} = \frac{2}{3}(5^3 - 3^3)$$

As a decimal the answer is approximately 65.3333.
Correct Answers:

- `65.3333`

Problem 2.
3. **(5 points)** `local/rmb-problems/e3/volume-shells-mc.pg`

A solid is formed by rotating the region enclosed by the curves $y = x^3$, $y = 0$, $x = 1$, and $x = 2$ about the y-axis. Select the integral which computes the resulting volume.

- A. $2\pi \int_1^2 x^4\,dx$
- B. $2\pi \int_1^2 x\sqrt{1+9x^4}\,dx$
- C. $2\pi \int_0^1 x^4\,dx$
- D. $\pi \int_1^2 x^6\,dx$
- E. $\pi \int_0^1 x^6\,dx$

Solution: *(Instructor solution preview: show the student solution after due date.)*

SOLUTION
If fix x and rotate the line segment from $(x,0)$ to (x,x^3) about the y-axis, we obtain a shell with with height $h = x^3$ and radius $r = x$. The total volume will be $= 2\pi \int_1^2 rh\,dx = 2\pi \int_1^2 x^4\,dx$.
Correct Answers:

- `A`

Problem 3.
6. **(5 points)** `local/rmb-problems/e3/surface-area-2-mc.pg`

The graph of $f(x) = x^2$ between the points $(2,4)$ and $(3,9)$ is rotated about the x-axis. Select the integral which computes the area of the resulting surface.

- A. $2\pi \int_2^3 x\sqrt{1+4x^2}\,dx$
- B. $2\pi \int_4^9 x^2\sqrt{1+x^4}\,dx$
- C. $2\pi \int_2^3 x^2\sqrt{1+4x^2}\,dx$
- D. $2\pi \int_4^9 x\sqrt{1+x^4}\,dx$
- E. $2\pi \int_2^3 x\sqrt{1+x^4}\,dx$

Solution: *(Instructor solution preview: show the student solution after due date.)*

SOLUTION
The differential of arc-length along the curve (x,x^2) is $ds = \sqrt{1+4x^2}\,dx$ and the curve lies between $x = 2$ and $x = 3$. If we rotate the point (x,x^2) about the x-axis we obtain a circle of radius $r = x^2$. The surface area of the resulting surface is $2\pi \int_2^3 r\,ds = 2\pi \int_2^3 x^2\sqrt{1+4x^2}\,dx$.
Correct Answers:

- `C`

Problem 4.
8. **(5 points)** `local/rmb-problems/e3/center-of-mass-num.pg`

Three equal masses are placed at the points $(-4,-3)$, $(4,-3,)$, and $(0,3)$. Find the coordinates $(\bar{x},\bar{y})$ of the center of mass.
$\bar{x} =$ ____, $\bar{y} =$ ____.

Exact answers are preferred. Your answer should be correctly rounded to three decimal places, or more accurate.

Solution: *(Instructor solution preview: show the student solution after due date.)*

SOLUTION

If m is the mass, then total mass $M = 3m$. The moment about the y-axis is $M_y = -4m + 4m + 0m = 0$.
The moment about the x-axis is $M_x = 3m + 2 \cdot (-3)m = -3m$.
Thus we have the center of mass is $(\bar{x}, \bar{y}) = (M_y/M, M_x/M) = (0, -1)$.

Correct Answers:

- `0`
- `-1`

Problem 5.

4. **(5 points)** `local/rmb-problems/e3/washers-2-mc.pg`

Let T be the triangle that is enclosed by the lines with equations $y = x$, $y = 2x - 1$ and $x = 3$. We rotate the triangle T about the x-axis to obtain a solid of rotation S. Which of the following integrals computes the volume of the solid S?

- A. $\pi \int_1^3 ((2x-1)^2 - 3^2)\, dx$
- B. $\pi \int_1^3 (3^2 - x^2)\, dx$
- C. $\pi \int_1^3 ((2x-1)^2 - x^2)\, dx$
- D. $\pi \int_1^3 (x-1)^2\, dx$
- E. $\pi \int_1^5 ((2x-1)^2 - x^2)\, dx$

Solution: *(Instructor solution preview: show the student solution after due date.)*

SOLUTION

We solve $2x - 1 = x$ to find that the lines $y = 2x - 1$ and $y = x$ intersect at $x = 1$. Thus the triangle T will lie between $x = 1$ and $x = 3$. The line $y = 2x - 1$ lies above the line $y = x$ for $1 \le x \le 3$. If we intersect the solid of revolution S with a plane which passes through x and is perpendicular to the x-axis, we obtain a washer with inner radius x and outer radius $2x - 1$. The area of this washer is $A(x) = \pi((2x-1)^2 - x^2)$. Integrating, we find the volume is

$$\pi \int_1^3 ((2x-1)^2 - x^2)\, dx.$$

Correct Answers:

- `C`

Problem 6.

2. **(5 points)** `local/rmb-problems/e3/vol-slice-num.pg`

A solid lies between $x = 2$ and $x = 5$. The cross-section at x is a circle with radius $r = 7x^2$. Find the volume of the solid.
The volume is __________

Exact answers are preferred. Your answer should be correctly rounded to three decimal places, or more accurate.

Solution: *(Instructor solution preview: show the student solution after due date.)*

SOLUTION

The area of the cross-section at x is $A(x) = \pi r^2 = \pi 7^2 x^4$. The volume is $\int_2^5 A(x)\, dx$. Evaluate this integral gives the volume as

$$\pi 7^2 (5^5/5 - 2^5/5).$$

Evaluating this as a decimal gives the volume as approximately 95226.1.

Correct Answers:

- `pi*7^2*(5^5/5-2^5/5)`

Problem 7.

7. **(5 points)** `local/rmb-problems/e3/moment-mc.pg`

Which of the following integrals represents the y-moment M_y of a thin plate that covers the region enclosed by the graphs $f(x) = x^2 - 4x + 6$ and $g(x) = x + 2$? The density of the plate is $\rho = 3$.

- A. $M_y = \int_1^4 (-x^2 + 5x - 4)\, dx$
- B. $M_y = 3 \int_1^4 x(-x^2 + 5x - 4)\, dx$
- C. $M_y = 3 \int_1^4 (-x^2 + 5x - 4)\, dx$
- D. $M_y = \frac{3}{2} \int_1^4 ((2+x)^2 - (x^2 - 4x + 6)^2)\, dx$
- E. $M_y = 3 \int_1^4 x(-x^2 + 3x - 8)\, dx$

Solution: *(Instructor solution preview: show the student solution after due date.)*

SOLUTION

Solving the equation $x + 2 = x^2 - 4x + 6$, we find the graphs intersect at $x = 1$ and $x = 4$. The function $x + 2$ is larger than $x^2 - 4x + 6$ in this interval.

If we take a thin strip of the plate at x with width dx, the strip is x units from the y-axis and has height $(x+2) - (x^2 - 4x + 6) = -x^2 + 5x - 4$. The area is $(-x^2 + 5x - 4)\, dx$ and we multiply this area by the density to obtain the mass. Next, multiplying by the distance to the y-axis gives that the moment of this strip is $3x(-x^2 + 5x - 4)\, dx$. Integrating this expression from 1 to 4 gives

$$M_y = 3 \int_1^4 x(-x^2 + 5x - 4)\, dx.$$

Correct Answers:

- `B`

Problem 8.

1. **(5 points)** `local/rmb-problems/e3/average-num.pg`

Find the average value of the function $\sec^2(x)$ on the interval $[-\pi/6, \pi/4]$.
The average value is ______

Exact answers are preferred. Your answer should be correctly rounded to three decimal places, or more accurate.

Solution: *(Instructor solution preview: show the student solution after due date.)*

SOLUTION

Since the anti-derivative of $\sec^2(x)$ is $\tan(x)$, we have

$$\int_{-\pi/6}^{\pi/4} \sec^2(x)\,dx = \tan(x)\big|_{x=-\pi/6}^{\pi/4} = \tan(\pi/4) - \tan(-\pi/6).$$

To find the average value we need to divide by the length of the interval $\pi/4 + \pi/6$ to find the answer

$$\frac{\tan(\pi/4) - \tan(-\pi/6)}{\pi/4 + \pi/6}$$

Evaluating this expression as a decimal gives an answer of approximately 1.20501.

Correct Answers:

- `[tan(pi/4)-tan(-pi/6)]/(pi/6+pi/4)`

This is the free response part of Exam 3. There are 3 questions, each worth 20 points. Please write your solutions in full, clearly indicating each step leading to the final answer. Omitting details will result in a lower grade.

$\boxed{10}$ **Question 1.** (a) Find the average value f_{ave} of the function $f(x) = \sin^2(x)$ on the interval $[0, \pi]$.

SOLUTION: We have

$$\underbrace{f_{\text{ave}} = \frac{1}{\pi}\int_0^\pi \sin^2 x\,dx}_{5} = \underbrace{\frac{1}{\pi}\int_0^\pi \frac{1-\cos 2x}{2}\,dx}_{\text{proper int. method } 4} = \underbrace{\frac{1}{\pi}\left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^\pi = \frac{1}{2}}_{1}.$$

$\boxed{10}$ (b) Find **all** the values c in $[0, \pi]$ satisfying $f(c) = f_{\text{ave}}$.

SOLUTION: Writing

$$\sin^2 c = \frac{1}{2},$$

and taking into account that $\sin x \geq 0$ in $[0, \pi]$, we have

$$\sin c = \frac{\sqrt{2}}{2},$$

proper setup 5

and

$$\underbrace{c = \frac{\pi}{4} \qquad \text{or} \qquad c = \frac{3\pi}{4}}_{5}.$$

★

If only one c found – 3

20 **Question 2.** Let $\mathcal{R}$ be the part of the disk $x^2+y^2 \leq 4$ that lies above the line $y=1$. Find the volume of the solid of revolution $\mathcal{S}$ obtained by rotating $\mathcal{R}$ about the x-axis. Clearly state which method (washer or cylindrical shells) you are using.

SOLUTION: (washer method) The region $\mathcal{R}$ is bounded by the curves $y=1$ and $y=\sqrt{4-x^2}$ which intersect where

$$\sqrt{4-x^2}=1,$$

i.e. for $x=\pm\sqrt{3}$. Using the washer method, we have

$$\underbrace{\mathrm{Vol}(\mathcal{S}) = \pi \int_{-\sqrt{3}}^{\sqrt{3}} [(4-x^2)-1]\,dx}_{10} = \underbrace{2\pi \int_0^{\sqrt{3}} (3-x^2)\,dx = 2\pi \left[3x - \frac{x^3}{3}\right]_0^{\sqrt{3}} = 4\sqrt{3}\,\pi.}_{10}$$

★

SOLUTION: (cylindrical shells method) We have, using the substitution $u=4-y^2$,

$$\begin{aligned}
\mathrm{Vol}(\mathcal{S}) &= \int_1^2 (2\pi y)(2\sqrt{4-y^2})\,dy \quad \} \ 10 \\
&= -2\pi \int_3^0 \sqrt{u}\,du \\
&= 2\pi \int_0^3 \sqrt{u}\,du \\
&= \left[\frac{4\pi}{3}u^{3/2}\right]_0^3 \\
&= 4\pi\sqrt{3}.
\end{aligned}$$

10

★

Either method is acceptable.

For both methods:

- proper setup of the integral – 10
- proper evaluation of the integral – 10

(next page)

20

Question 3. Find the centroid of the region in the first quadrant of the xy-plane bounded by the curves $y = x^3$ and $x = y^3$.

SOLUTION: Let $(\bar{x}, \bar{y})$ be the centroid. Since the region is symmetric about the line $y = x$, the symmetry principle implies that $\bar{y} = \bar{x}$. First, compute the area of the region:

$$\left. A = \int_0^1 (x^{1/3} - x^3)\, dx = \left[\frac{3x^{4/3}}{4} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2}. \right\} \; 6$$

Then

$$6 \quad \underbrace{\bar{x} = 2\int_0^1 x(x^{1/3} - x^3)\, dx}_{} = \underbrace{2\int_0^1 (x^{4/3} - x^4)\, dx = 2\left[\frac{3x^{7/3}}{7} - \frac{x^5}{5}\right]_0^1 = \frac{16}{35}}_{6}.$$

Thus the centroid is (16/35, 16/35). 2 ★

Note that without using the symmetry principle, one can compute $\bar{y}$ directly:

$$\underbrace{\bar{y} = 2\int_0^1 \frac{1}{2}(x^{2/3} - x^6)\, dx}_{6} = \underbrace{\left[\frac{3x^{5/3}}{5} - \frac{x^7}{7}\right]_0^1 = \frac{16}{35}}_{6},$$

as expected.

(a) Correct A − 6

(b) Correct setup for $\bar{x}$ or $\bar{y}$ − 6
[Ignore possible confusion $\bar{x} \leftrightarrow \bar{y}$]

(c) Correct computation of the integral in (b) − 6

(d) Correct conclusion that $\bar{x} = \bar{y} = 16/35$
either using symmetry or by computation
of both $\bar{x}$ and $\bar{y}$ − 2

(end of exam questions)