EXAM 4

Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1 (A) (B) (C) (D) (E) **6** (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E) **7** (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E) **8** (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E) **9** (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E) **10** (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

Trig identities

- $\sin^2(x) + \cos^2(x) = 1$,
- $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ and $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
- $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ and $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

Multiple Choice Questions

1. (5 points) Find the volume of a solid whose base is the unit circle $x^2 + y^2 = 1$ and the cross sections perpendicular to the x-axis are squares.

   A. 0
   B. $\frac{8}{3}$.
   C. $\frac{16}{3}$.
   D. $\frac{4}{3}$.
   E. $\frac{32}{3}$.

2. (5 points) Which of the following is the equation for a ellipse with vertices $(\pm 5, 0)$ and foci $(\pm 4, 0)$?

   A. $\frac{x^2}{5} - \frac{y^2}{3} = 1$.
   B. $\frac{x^2}{25} + \frac{y^2}{9} = 1$.
   C. $\frac{y^2}{25} + \frac{x^2}{9} = 1$.
   D. $\frac{y^2}{9} - \frac{x^2}{16} = 1$.
   E. $\frac{x^2}{9} + \frac{y^2}{16} = 1$.

3. (5 points) Find the Taylor polynomial $T_3(x)$ for $\sin(2x)$ centered at 0. What is $T_3(1)$?

   A. $2 - \frac{4}{3}$**.**

   B. $1 + 3 + \frac{1}{2} + \frac{7}{6}$.

   C. $1 - 3 - \frac{1}{2} - \frac{4}{3}$.

   D. $3 - \frac{4}{3}$.

   E. $1 - \frac{1}{6}$.

4. (5 points) Let $f(x) = \sqrt{x-1}$. Find a value of $c \in [2, 5]$ so that $f(c)$ is the average value of $f(x)$ on the interval $[2, 5]$.

   A. $\frac{14}{9} - 1$.

   B. $\left(\frac{5}{3}\right)^2 + 1$.

   C. $\left(\frac{21}{9}\right)^2$.

   D. $\left(\frac{14}{9}\right)^2 + 1$**.**

   E. $\frac{14}{9}$.

5. (5 points) Evaluate $\int_0^\infty x^3 e^{-x^4}\, dx$

   A. 0.

   B. 1/4**.**

   C. 4.

   D. $-1/4$.

   E. This integral diverges..

6. (5 points) Use Simpson's Rule with $n = 4$ intervals to approximate $\int_0^2 \sqrt{1+4x^2}\,dx$.

   A. $\frac{1}{2}(\frac{\sqrt{20}}{4} + \frac{\sqrt{52}}{4} + \frac{\sqrt{116}}{4} + \frac{\sqrt{212}}{4})$.

   B. $\frac{1}{3}(1 + 2\sqrt{2} + 4\sqrt{5} + 2\sqrt{10} + \sqrt{17})$.

   C. $\frac{1}{6}(1 + 2\sqrt{2} + 4\sqrt{5} + 2\sqrt{10} + \sqrt{17})$.

   D. $\frac{1}{6}(1 + 4\sqrt{2} + 2\sqrt{5} + 4\sqrt{10} + \sqrt{17})$**.**

   E. $\frac{1}{3}(1 + 4\sqrt{2} + 2\sqrt{5} + 4\sqrt{10} + \sqrt{17})$.

7. (5 points) A surface is created by rotating the graph of $f(x) = 1 + \sin(x)$ from $x = 0$ to $x = 100$ around the x-axis. What is the integral that computes the area of this surface?

   A. $\int_0^{100} 2\pi(1 + \sin(x))\sqrt{1 + (\cos(x))^2}dx$**.**

   B. $\int_0^{100} 2\pi(1 + \sin(x))dx$.

   C. $\int_0^{100} 2\pi\sqrt{1 + (\cos(x))^2}dx$.

   D. $\int_0^{100} 2\pi(1 + \sin(x))\sqrt{1 + \cos(x)}dx$.

   E. $\int_0^{100} 2\pi(1 + \cos(x))\sqrt{1 + (\sin(x))^2}dx$.

8. (5 points) Find the center of mass of the region between the curves $y = 1 - x^2$ and $y = x^2$ (Assume the region has constant density).

   A. $\left(\frac{1}{2}, \frac{1}{\sqrt{2} - \frac{2}{3}(\sqrt{2})^3}\right)$.

   B. $\left(0, \sqrt{2} - \frac{2}{3}(\sqrt{2})^3\right)$.

   C. $\left(0, \frac{1}{2}\right)$**.**

   D. $\left(0, \frac{1}{\sqrt{2} - \frac{2}{3}(\sqrt{2})^3}\right)$.

   E. $\left(0, \frac{1}{\sqrt{2}}\right)$.

9. (5 points) Which of the following sequences converge?

A. $b_n = \frac{5^n}{n!}$.

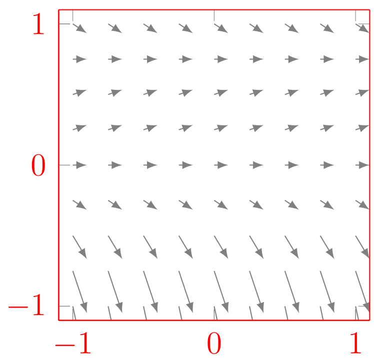
B. $c_n = \frac{2n + (-1)^n}{n}$.

C. $a_n = \frac{n(1-3n)}{n^3+1}$.

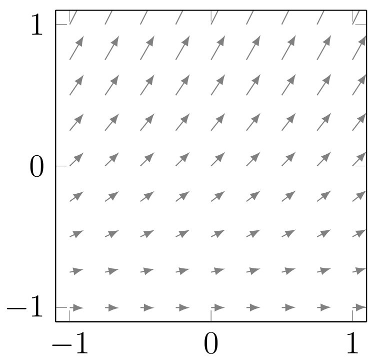
D. None of the above.

E. All of the above.

10. (5 points) Which of the following is the direction field for the equation $y' = 2y - \frac{8}{3}y^2$?



A.



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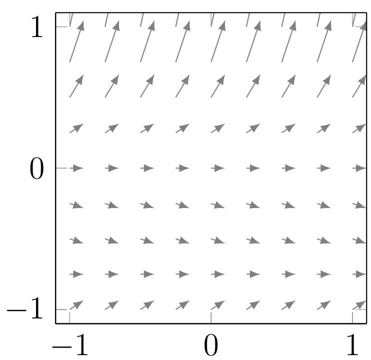

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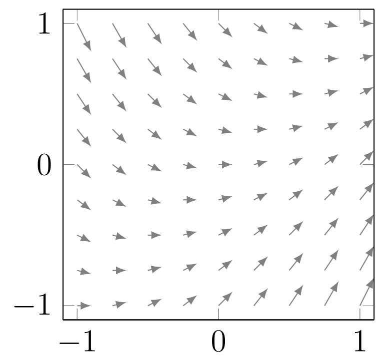

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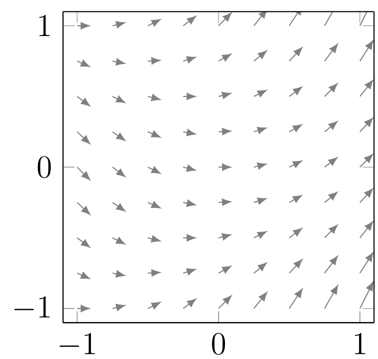


E.

Free Response Questions

11. (a) (5 points) Compute the following integral.

$$\int \frac{dx}{x^2\sqrt{x^2-\pi}}.$$

Solution: Take $x = \sqrt{\pi}\sec\theta$. Then $dx = \sqrt{\pi}\sec\theta\tan\theta d\theta$.

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2-\pi}} &= \int \frac{\sqrt{\pi}\sec\theta\tan\theta d\theta}{(\sqrt{\pi}\sec\theta)^2\sqrt{(\sqrt{\pi}\sec\theta)^2-\pi}} \\ &= \int \frac{\sqrt{\pi}\sec\theta\tan\theta d\theta}{\pi\sec^2\theta\sqrt{\pi\sec^2\theta-\pi}} = \int \frac{\tan\theta d\theta}{\pi\sec\theta\tan\theta} = \frac{1}{\pi}\int\cos\theta d\theta \\ &= \frac{1}{\pi}\sin\theta + C = \frac{1}{\pi}\frac{\sqrt{x^2-2}}{x} + C\end{aligned}$$

(b) (5 points) Determine if the following series converges or diverges.

$\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3n}{\sqrt{5+n^5}}$. Justify your answer!

Solution: Compute

$$\lim_{n\to\infty} \frac{\frac{n^2+3n}{\sqrt{5+n^5}}}{\frac{1}{n^{1/2}}} = \lim_{n\to\infty} \frac{(n^2+3n)n^{1/2}}{\sqrt{5+n^5}} = \lim_{n\to\infty} \frac{n^{\frac{5}{2}}+3n^{\frac{3}{2}}}{\sqrt{5+n^5}} = 1.$$

Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ diverges this series diverges by the limit comparison test.

12. (a) (5 points) Set up an integral for the arc length of the hyperbola $xy = 1$ from $x = 1$ to $x = 2$.

Solution:

$$L = \int_1^2 \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \int_1^2 \sqrt{1 + \frac{1}{x^4}}\, dx$$

(b) (5 points) Use Simpson's Rule with $n = 4$ to estimate the arc length.

Solution: Let $f(x) = \sqrt{1 + \frac{1}{x^4}}$. Then

$$S_4 = \frac{0.25}{3}(f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2)) \approx 1.13254$$

13. Let S be the solid obtained by rotating the region bounded by the circle $x^2 + y^2 = 1$ around the line $x = 3$.

(a) (3 points) Set up the integral that computes the volume of S using the disk/washer method.

Solution:

$$\pi \int_{-1}^{1} (3 + \sqrt{1 - y^2})^2 - (3 - \sqrt{1 - y^2})^2 dy$$

(b) (3 points) Set up the integral that computes the volume of S using the cylindrical shells method.

Solution:

$$\int_{-1}^{1} 2\pi(3 - x)(2\sqrt{1 - x^2})dx$$

(c) (4 points) Choose one of these integrals and find the volume of S.

Solution: With the integral in (a)

$$\pi \int_{-1}^{1} (3 + \sqrt{1 - y^2})^2 - (3 - \sqrt{1 - y^2})^2 dy =$$

$$\pi \int_{-1}^{1} (9+6\sqrt{1 - y^2}+(1-y^2))-(9-6\sqrt{1 - y^2}+(1-y^2))dy = 12\pi \int_{-1}^{1} \sqrt{1 - y^2}dy.$$

We do a trig substitution with $y = \sin(\theta)$, $dy = \cos(\theta)d\theta$ to get:

$$12\pi \int \cos^2(\theta)d\theta = 6\pi \int (1+\cos(2\theta))d\theta = 6\pi[\theta+\frac{1}{2}\sin(2\theta)] = 6\pi[\theta+\cos(\theta)\sin(\theta)].$$

Now $\theta = \arcsin(y)$, so $6\pi[\arcsin(y)+y\sqrt{1 - y^2}]_{-1}^{1} = 6\pi[\arcsin(1)-\arcsin(-1)] = 6\pi[\frac{\pi}{2} - (-\frac{\pi}{2})] = 6\pi^2$.

Solution: With the integral in (b)

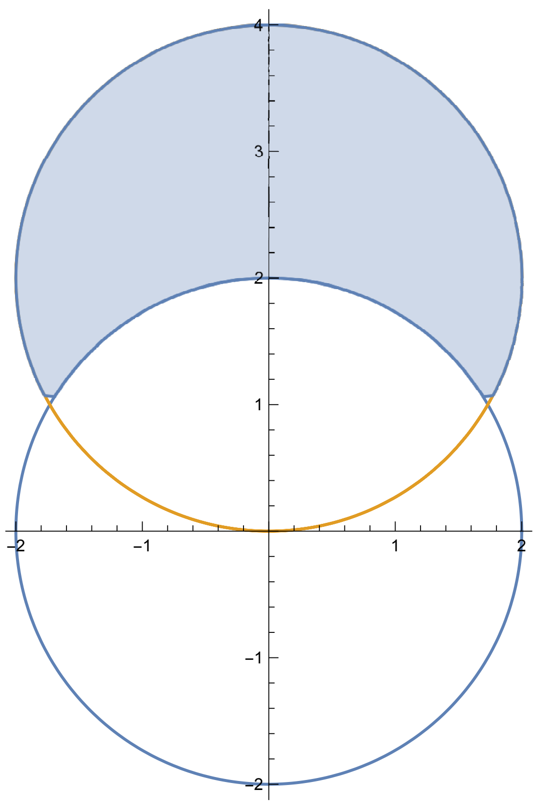
$$\int_{-1}^{1} 2\pi(3 - x)(2\sqrt{1 - x^2})dx = 12\pi \int_{-1}^{1} \sqrt{1 - x^2}dx - 4\pi \int_{-1}^{1} x\sqrt{1 - x^2}dx$$

First, for $4\pi \int_{-1}^{1} x\sqrt{1 - x^2}dx$ we do a substitution with $u = 1-x^2$, $du = -2xdx$, giving $xdx = -\frac{1}{2}du$. This gives $-2\pi \int u^{\frac{1}{2}}du = -\frac{4\pi}{3}u^{\frac{3}{2}}$, so we get $-\frac{4\pi}{3}[(1 - x^2)^{\frac{3}{2}}]_{-1}^{1} = -\frac{4\pi}{3}[0-0] = 0$. Now $12\pi \int_{-1}^{1} \sqrt{1 - x^2}dx$ is the same integral as in the solution to (a), so = we get $6\pi^2 - 0 = 6\pi^2$.

14. (a) (5 points) Find the slope of the tangent line to the circle $r = 4\sin\theta$ at the point $(\sqrt{3}, 1)$.

Solution: We can parametrize this circle by $x(\theta) = 4\sin\theta\cos\theta$, $y(\theta) = 4\sin^2\theta$, with $0 \leq \theta \leq \pi$.
$4\sin^2\theta = 1$, so $\sin\theta = \frac{1}{2}$, so $\theta = \frac{\pi}{6}$.
The slope is given by $\frac{y'(\theta)}{x'(\theta)} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$, which is $\tan(2\frac{\pi}{6}) = \sqrt{3}$.

(b) (5 points) Calculate the area of the region (shown below) that is outside the polar curve $r = 2$ and inside the polar curve $r = 4\sin\theta$.



Solution: The curves intersect when

$$2 = 4\sin\theta$$
$$\sin\theta = \frac{1}{2}$$

so $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$. Then the area is given by the integral

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}((4\sin\theta)^2 - (2)^2)d\theta = \frac{4}{3}\pi + 2\sqrt{3}$$

15. (a) (5 points) Find the solution to the differential equation $\frac{dy}{dx} = x(1-y)$ that satisfies the initial condition $y(0) = -1$. Your solution should be an expression of y in terms of x.

Solution:

$$y = 1 - 2e^{-\frac{1}{2}x^2}$$

(b) (5 points) Find the limit of your solution in (a) as $x \to \infty$. That is, find $\lim_{x\to\infty} y(x)$.

Solution:

$$\lim_{x\to\infty} y(x) = 1.$$