Answer all of the questions 1 - 7 and two of the questions 8 - 10. Please indicate which problem is not to be graded by crossing through its number in the table below.

Additional sheets are available if necessary. No books or notes may be used. Please, turn off your cell phones and do not wear ear-plugs during the exam. You may use a calculator, but not one which has symbolic manipulation capabilities. Please:

1. clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
2. give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question.

Name: ______________________ Answer Key

Section: ____________

Last four digits of student identification number: ____________

Question	Score	Total
1		13
2		10
3		8
4		8
5		8
6		9
7		9
8		16
9		16
10		16
Free	3	3
		100

(1) Calculate the following limits or show that they do not exist. If the limit in question does not exist, but is ∞ or $-\infty$, then clearly indicate that.

(3)

(a) $\lim_{x\to 4} \frac{x^2-5x+4}{x^2-3x-4}$ $\overset{(1)}{=} \lim_{x\to 4} \frac{\cancel{(x-4)}(x-1)}{\cancel{(x-4)}(x+1)} \overset{(1)}{=} \lim_{x\to 4} \frac{x-1}{x+1} \overset{(1)}{=} \frac{3}{5}$,

where the last step follows because $\frac{x-1}{x+1}$ is continuous at 4.

(2)

(b) $\lim_{x\to 4} \frac{x^2+5x+4}{x^2+3x-4}$ $\overset{(1)}{=} \frac{40}{24} \overset{(1)}{=} \frac{5}{3}$

because this rational function is continuous at 4.

(4)

(c) $\lim_{x\to 3} \frac{5-x}{(x-3)^2}$ $\overset{(1)}{=} +\infty$

$5-x \xrightarrow{x\to 3} 2$ (1)

$(x-3)^2 \xrightarrow{x\to 3} 0$ (1)

and $(x-3)^2 > 0$ ~~for all x~~ (1)

2 points for answer DNE

(4)

(d) $\lim_{h\to 0} \frac{(h-1)^2-1}{h}$. $\overset{(1)}{=} \lim_{h\to 0} \frac{h^2-2h+1-1}{h} \overset{(1)}{=} \lim_{h\to 0} \frac{h^2-2h}{h}$

$\underset{h\neq 0}{\overset{(1)}{=}} \lim_{h\to 0} h-2 \overset{(1)}{=} -2$, again by direct substitution because $h-2$ is continuous.

(a) $\frac{3}{5}$

(b) $\frac{5}{3}$

(c) $+\infty$

(d) -2

(2) (a) Solve the equation $2^{x-7} = 3$. Give the exact answer.

6

2 [$\log_2(2^{x-7}) = \log_2(3)$

2 [$x - 7 = \log_2(3)$

2 [$x = \underline{\underline{\log_2(3) + 7}}$

Any exact answer using other logarithms gets full credit. A decimal approximation to the exact answer gets 3 points.

(b) Express the quantity

$$\log_3(1 + x^2) + \frac{1}{2}\log_3(x) - \log_3(4x)$$

as a single logarithm.

4

$\log_3(1+x^2) + \frac{1}{2}\log_3(x) - \log_3(4x)$

$= \log_3(1+x^2) + \log_3(x^{\frac{1}{2}}) - \log_3(4x)$

$= \log_3\left(\frac{(1+x^2)\sqrt{x}}{4x}\right)$

by using the laws of logarithms.

(a) $\log_2(3) + 7$

(b) $\log_3\left(\frac{(1+x^2)\sqrt{x}}{4x}\right)$

(3) Consider the functions $f(x) = \sqrt{10 - x}$ and $g(x) = x^2+1$. Let h be the composite function $h(x) = (f \circ g)(x)$.

(a) Compute $h(2)$.

2 [$h(x) = \sqrt{10-(x^2+1)} = \sqrt{9-x^2}$

1 [$h(2) = \sqrt{9-2^2} = \underline{\underline{\sqrt{5}}}$

(b) Find the domain of h. **As usual, justify your answer.**

2 [$h(x)$ is defined for all x for which $9 - x^2 \geq 0$.

1 [thus $x^2 \leq 9$ and this

1 [means $-3 \leq x \leq 3$.

1 [Hence the domain is $[-3, 3]$

(a) $h(2) =$ $\sqrt{5}$

(b) Domain of h is $[-3, 3]$

(4) Let f be a function such that, for all real numbers x near 5,

$$\frac{1}{5}x + \frac{5}{x} + 2 \leq f(x) \leq x^2 - 10x + 29.$$

Argue that $\lim_{x\to 5} f(x)$ exists and find its value. As usual, justify your answer.

Since polynomial and rational functions are continuous on their domain, we get

2 $$\lim_{x\to 5}\left(\frac{1}{5}x + \frac{5}{x} + 2\right) = \frac{1}{5}\cdot 5 + \frac{5}{5} + 2 = 4$$

2 $$\lim_{x\to 5}\left(x^2 - 10x + 29\right) = 25 - 50 + 29 = 4$$

2 Since both limits agree, we may use the squeeze theorem and

2 conclude

$$\lim_{x\to 5} f(x) = 4.$$

$\lim_{x\to 5} f(x) =$ 4

(5) Let f and g be two functions such that the following limits exist

$$\lim_{x\to 3} g(x) = 6, \quad \lim_{x\to 3}\left[xf(x) - 2^x g(x)\right] = 12.$$

Use the limit laws to compute the following limits.

3

(a) $\lim_{x\to 3} \frac{x^2-4}{g(x)}$. $\underset{\text{Limit Law}}{=} \frac{\lim_{x\to 3}(x^2-4)}{\lim_{x\to 3} g(x)} \underset{\text{Polynomials are continuous}}{=} \frac{3^2-4}{6} = \frac{5}{6}$

(b) $\lim_{x\to 3} f(x)$.

Set $L = \lim_{x\to 3} f(x)$. Then

2

$\lim_{x\to 3}\left(x f(x) - 2^x g(x)\right)$

$\underset{\text{limit laws}}{=} \left(\lim_{x\to 3} x\right)\cdot\left(\lim_{x\to 3} f(x)\right) - \left(\lim_{x\to 3} 2^x\right)\left(\lim_{x\to 3} g(x)\right)$

$= 3\cdot L - 2^3\cdot 6 = 3\cdot L - 48$

3

Hence we know that $3\cdot L - 48 = 12$ and thus $3L = 60$. Hence

$L = \lim_{x\to 3} f(x) = 20$

(a) $\frac{5}{6}$

(b) 20

(6) Consider the function

$$f(x) = \frac{2x-1}{4x-6}.$$

(a) Find the domain of f.

2 [We need $4x-6 \neq 0$ for $f(x)$ to be defined, hence $x \neq \frac{6}{4} = \frac{3}{2}$ and

1 [the domain of f is
$\{x \mid x \neq \frac{3}{2}\}$

Set notation or interval notation for domain required.

(b) Find the inverse function f^{-1} of f.

2 [Set $y = \frac{2x-1}{4x-6}$

and solve for x:

3 [
$$4yx - 6y = 2x - 1$$
$$(4y-2)x = 6y - 1$$
$$x = \frac{6y-1}{4y-2}$$

1 [Hence $f^{-1}(x) = \frac{6x-1}{4x-2}$

(a) Domain of f is $\{x \mid x \neq \frac{3}{2}\}$

(b) $f^{-1}(x) = \frac{6x-1}{4x-2}$

(7) An object is moving on a straight line so that at time t seconds it is located at

$$s(t) = t^2 + 5t$$

meters to the right of some reference point. In the following problems also give the units with your answers.

(a) Find the average velocity of the object for the time interval $2 \le t \le 4$.

2

$$\frac{s(4)-s(2)}{4-2} = \frac{36-14}{2} = \frac{22}{2} = 11 \text{ m/sec.}$$

(b) Find the average velocity for the time interval $[2, t]$, where $t > 2$. **Simplify your answer.**

4

$$\frac{s(t)-s(2)}{t-2} \overset{1}{=} \frac{t^2+5t-14}{t-2} \overset{1}{=} \frac{(t+7)(t-2)}{(t-2)} \underset{t\neq 2}{\overset{1}{=}} t+7 \text{ m/sec.}$$

1 point deduction for forgetting the units once or twice. 2 points deductions for no units at all.

(c) Use your answer in (b) to find the instantaneous velocity of the object at time $t = 2$. 1

3

$$\underbrace{\lim_{t\to 2} \frac{s(t)-s(2)}{t-2}}_{1} \underset{(b)}{=} \lim_{t\to 2}(t+7) = 9 \text{ m/sec}$$

1 $t+7$ is continuous.

(a) Average velocity over [2, 4] is 11 meters/seconds

(b) Average velocity over [2, t] is $t+7$ meters/seconds

(c) Instantaneous velocity at time $t = 2$ is 9 meters/seconds

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(8) (a) Define what it means for a function f to be continuous at a. Use complete sentences.

4 A function f is continuous at a if $\lim_{x\to a} f(x) = f(a)$.

Let

$$f(x) = \begin{cases} cx^2 - 2x + 9, & \text{if } x < 2, \\ 8, & \text{if } x = 2, \\ \frac{4}{x} + 3c, & \text{if } x > 2. \end{cases}$$

For the following problems, always justify your answer!

(b) Find all values for c such that $\lim_{x\to 2} f(x)$ exists.

1 Since cx^2-2x+9 and $\frac{4}{x}+3c$ are continuous at $x \neq 2$ we get

2 $\lim_{x\to 2^-} f(x) = \lim_{x\to 2}(cx^2-2x+9) = 4c+5$ and

2 $\lim_{x\to 2^+} f(x) = \lim_{x\to 2}(\frac{4}{x}+3c) = 2+3c$. For the

2 two-sided limit to exist, we need $4c+5 = 2+3c$, that is, $c=-3$.

(c) For which of the values for c found in (b) is the function f continuous at 2?

not needed For f to be continuous at 2, the two-sided limit at 2 has to exist. By (b) this forces $c=-3$. For $c=-3$ we have

3 $\lim_{x\to 2} f(x) = 2-9 = -7$. But $f(2)=8$. Therefore, f is not continuous at 2 for $c=-3$.

(d) Find all values for c such that the function f is continuous at 0.

not required Since polynomials are continuous everywhere we have

2 $\lim_{x\to 0} f(x) = \lim_{x\to 0}(cx^2-2x+9) = 9 = f(0)$ for all numbers c and f is continuous.

(b) -3 (c) none. (d) $\mathbb{R}$

(9) (a) State the Intermediate Value Theorem. Use complete sentences.

6 Let f be a continuous function on the closed interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a,b) such that $f(c) = N$.

(b) Explain in detail why and how you can use this theorem to show that the equation

$$x^4 - 5x^2 = 2$$

has a solution in the interval (2, 3).

1 Consider $f(x) = x^4 - 5x^2 - 2$.

1 This function is continuous everywhere and in particular on $[2,3]$.

3 We have

$f(2) = 16 - 20 - 2 = -6$

$f(3) = 81 - 45 - 2 = 34$.

2 We may choose $N = 0$, which is between $f(2)$ and $f(3)$.

2 Then the IVT tells us that there exists some c in $(2,3)$ such that $f(c) = N = 0$. Hence

1 $c^4 - 5c^2 = 2$

and c is the desired solution.

(10) (a) State the definition of the derivative of a function at a point a. Use complete sentences.

5

the derivative of f at a is defined as
$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\quad\left(\text{or } \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\right),$$
provided the limit exist.

Consider now the function
$$f(x)=\sqrt{x+8}.$$

(b) Compute the slope of the secant line through the points $(1, f(1))$ and $(8, f(8))$.

2

$$\frac{f(8)-f(1)}{8-1}=\frac{\sqrt{16}-\sqrt{9}}{7}=\boxed{\frac{1}{7}}$$

(c) Compute the slope of the secant line through the points $(1, f(1))$ and $(1+h, f(1+h))$.

2

$$\frac{f(1+h)-f(1)}{h}=\frac{\sqrt{1+h+8}-3}{h}=\boxed{\frac{\sqrt{9+h}-3}{h}}$$

(d) Use part (c) to compute $f'(1)$.

4

$$f'(1)=\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h}=\lim_{h\to 0}\frac{(\sqrt{9+h}-3)(\sqrt{9+h}+3)}{h(\sqrt{9+h}+3)}$$
$$=\lim_{h\to 0}\frac{\cancel{9}+h-\cancel{9}}{h(\sqrt{9+h}+3)}=\lim_{h\to 0}\frac{1}{\sqrt{9+h}+3}=\boxed{\frac{1}{6}}$$
because rational functions, root functions and compositions thereof are continuous.

(e) Compute the equation of the tangent line to the graph of f at the point $(1, f(1))$. Put your answer in the form $y = mx + b$.

3

Point $(1, f(1)) = (1, 3)$

slope $= \frac{1}{6}$ from (d)

Hence $\quad y-3=\frac{1}{6}(x-1)$
$$\boxed{y=\frac{1}{6}x+\frac{17}{6}}$$