Worksheet # 6: Continuity

1. Comprehension check.

   (a) State and explain the intermediate value theorem.

   (b) Define what it means for $f(x)$ to be continuous at the point $x = a$. What does it mean if $f(x)$ is continuous on the interval $[a, b]$? What does it mean to say $f(x)$ is continuous?

   (c) There are three distinct ways in which a function will fail to be continuous at a point $x = a$. Describe the three types of discontinuity. Provide a sketch and an example of each type.

   (d) True or false? Every function is continuous on its domain.

   (e) True or false? The sum, difference, and product of continuous functions are all continuous.

   (f) If $f(x)$ is continuous at $x = a$, what can you say about $\lim_{x \to a^+} f(x)$?

   (g) Suppose $f(x), g(x)$ are continuous everywhere. What is $\lim_{x \to a} \frac{f(x)g(x) - f(x)^3}{g(x)^2 + 1}$?

2. Using the definition of continuity and properties of limits, show that the following functions are continuous at the given point a.

   (a) $f(x) = \pi$, $a = 1$

   (b) $f(x) = \frac{x^2 + 3x + 1}{x + 3}$, $a = -1$

   (c) $f(x) = \sqrt{x^2 - 9}$, $a = 4$.

3. Give the largest domain on which the following functions are continuous. Use interval notation.

   (a) $f(x) = \frac{x + 1}{x^2 + 4x + 3}$

   (b) $f(x) = \frac{x}{x^2 + 1}$

   (c) $f(x) = \sqrt{2x - 3} + x^2$

   (d) $f(x) = \begin{cases} x^2 + 1 & \text{if } x \leq 0 \\ x + 1 & \text{if } 0 < x < 2 \\ -(x - 2)^2 & \text{if } x \geq 2 \end{cases}$

4. State the intermediate value theorem and use the theorem to find an interval of length 1 in which a solution to the equation $2x^3 + x = 5$ must exist.

5. (Similar to MA 113 Exam I, problem 8, Spring 2009.) Let c be a number and consider the function

$$f(x) = \begin{cases} cx^2 - 5 & \text{if } x < 1 \\ 10 & \text{if } x = 1 \\ \frac{1}{x} - 2c & \text{if } x > 1 \end{cases}$$

   (a) Find all numbers c such that $\lim_{x \to 1} f(x)$ exists.

   (b) Is there a number c such that $f(x)$ is continuous at $x = 1$? Justify your answer.