Solution to Assignment 4 for MA 113 - Calculus I

(1) (6 points) Solve Exercise #69 in Section 3.5 (page 215) of the textbook.
Solution:
First we have to find the tangent line to the ellipse given by

$$x^2 + 4y^2 = 5 \tag{1}$$

that passes through the point $(-5, 0)$ and touches the ellipse at a point above the ground, that is, at a point whose y-coordinate is positive.
To this end let f be the function whose graph describes the above ellipse near a point (a, b) on the ellipse. Thus, f satisfies the equation

$$x^2 + 4\big(f(x)\big)^2 = 5.$$

Differentiating with respect to x we obtain

$$2x + 8f(x)f'(x) = 0. \tag{2}$$

Solving for $f'(x)$, we find that

$$f'(x) = -\frac{x}{4f(x)}.$$

Thus the equation of the tangent line to the ellipse at a point $(a, b) = (a, f(a))$ is given by

$$y - b = -\frac{a}{4b}(x - a). \tag{3}$$

This line passes through $(-5, 0)$ if and only if

$$-b = -\frac{a}{4b}(-5 - a).$$

Multiplying by $4b$ (see also the footnote on this page) we get

$$a^2 + 4b^2 + 5a = 0.$$

Since the point (a, b) lies on the ellipse, we also have that $a^2 + 4b^2 = 5$. If we substitute this, the previous equation becomes

$$5 + 5a = 0.$$

Hence, we obtain $a = -1$ and, using Equation (1), $(-1)^2+4b^2 = 5$, thus $b^2 = 1$. Since we look for a point of tangency with positive y-coordinate b, we conclude that $b = 1$. Using $(a, b) = (-1, 1)$ in Equation (3) gives the equation of the tangent line to the ellipse at $(-1, 1)$

$$y = \frac{1}{4}x + \frac{5}{4}.$$

The height of the lamp is the y-coordinate of the point on this line where $x = 3$. Hence the lamp is located $\frac{3}{4} + \frac{5}{4} = 2$ units above the x-axis.

Strictly speaking, we have to rule out the possibility $b = f(a) = 0$. However, if $b = 0$, then Equation (2) implies $a = 0$, but $(0, 0)$ is not a point on the ellipse. – Students are not expected to discuss this case.

(2) (4 Points) Find all tangent lines to the hyperbola $x^2 - 3y^2 = 18$ that pass through the point $(0, 2)$. Your solution should describe how you know that you have found all of the solutions.

Solution:

Let f be the function whose graph describes the given hyperbola near a point (a, b) on the hyperbola. Thus, f satisfies the equation

$$x^2 - 3\big(f(x)\big)^2 = 18. \tag{4}$$

Differentiating with respect to x we get $2x - 6f(x)f'(x) = 0$. Solving for $f'(x)$ we find

$$f'(x) = \frac{x}{3f(x)}.$$

Hence the equation of the tangent line to the hyperbola at $(a, b) = (a, f(a))$ is

$$y - b = \frac{a}{3b}(x - a). \tag{5}$$

This line passes through the point $(0, 2)$ if and only if

$$2 - b = \frac{a}{3b}(-a) = -\frac{a^2}{3b}.$$

Multiplying by $3b$ we obtain

$$6b + a^2 - 3b^2 = 0.$$

Since the point (a, b) is on the hyperbola, Equation (4) gives $a^2 - 3b^2 = 18$. If we substitute this, the previous equation becomes $6b + 18 = 0$, thus $b = -3$. Substituting $b = -3$ into Equation (4), we get $a^2 - 3(-3)^2 = 18$, thus $a = \sqrt{45}$ or $a = -\sqrt{45}$. The points $(a, b) = (\sqrt{45}, -3)$ and $(a, b) = (-\sqrt{45}, -3)$ both satisfy $a^2 - 3b^2 = 18$ and are therefore on the hyperbola. Using $(a, b) = (\sqrt{45}, -3)$ in Equation (5) gives the equation of the first tangent line

$$y = -\frac{\sqrt{45}}{9}x + 2,$$

and using $(a, b) = (-\sqrt{45}, -3)$ in Equation (5) provides the equation of the second tangent line

$$y = \frac{\sqrt{45}}{9}x + 2.$$

Bonus Problem (2 Points)

When Lola gets home after a long day at the university, she decides to run for a while. She begins running right outside her front door. First she runs on a level road, then she comes to a hill and runs to the top. When she gets to the top, she turns around and she runs back exactly the way she came. Now, on level ground, Lola can run at a speed of 10 km an hour. Uphill, she runs 7.5 km an hour and downhill 15 km an hour. Upon her return home, she notices that she had run for exactly two hours. How far did Lola run in total? (It seems that there is not enough information here to solve the problem, but there is.)

Solution: Let x be the distance between Lola's front door and her point of return. Denote by y the length of the uphill part of Lola's way. Since time is distance over velocity, Lola ran for $2\frac{x-y}{10}$ hours on level ground, $\frac{y}{7.5}$ hours uphill, and $\frac{y}{15}$ hours downhill. In total Lola ran for two hours, thus we get

$$2 = 2\frac{x-y}{10} + \frac{y}{7.5} + \frac{y}{15} = \frac{6(x-y) + 4y + 2y}{30} = \frac{6x}{30} = \frac{x}{5},$$

so $x = 10$. Hence Lola ran a whopping $2 \cdot 10 = 20$ km.

Grading Guidelines:

- Problem (1): (6 points) 1 point for realizing that the tangent line to the ellipse is needed, 1 point for the derivative, 2 points for the point of tangency, 1 point for the equation of the tangent line, and 1 point for the correct answer.

  Problem (2): (4 points) 1 point for the derivative, 2 points for finding the two points of tangency, and 1 point for the two equations of the tangent lines.

  Bonus Problem: (2 points) 1 point for the answer, 1 point for the reasoning.
- Please score in increments of at least 0.5 points. For each part, give full credit only for answers that are both *correct* and *fully explained*.
- Be sure to comment favorably on papers of students who do an unusually good job.
- Take the time to recognize and provide guidance to students who attempt unusual approaches.
- Deductions:
  i) If a student does not use complete sentences, mark with common error "EXP" and ask for complete sentences. Also mark common errors "ALG" and "EQN". Deduct one point for three or more such mistakes which are not otherwise penalized.

  ii) Deduct one point for unusually messy or poorly organized solutions (at most one or two per paper).