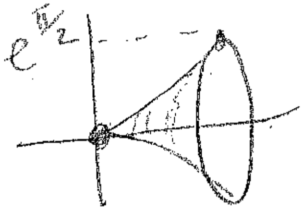
Free Response Questions: Show your work!

9. (16 points total) Find the surface area of the solid of revolution obtained by rotating the curve $c(t) = (e^t \cos t, e^t \sin t)$, for $t \in [0, \frac{\pi}{2}]$ about the x-axis.

$y(t) = e^t \sin t$



$$\times 2 \quad 2\pi \int_0^{\pi/2} y(t)\sqrt{x'(t)^2 + y'(t)^2}\,dt$$

$$\times 2 \quad x'(t) = e^t(\cos t - \sin t)$$
$$y'(t) = e^t(\sin t + \cos t)$$

$$x'(t)^2 + y'(t)^2 = e^{2t}\left[\cos^2 t - 2\cos t \sin t + \sin^2 t + \sin^2 t + 2\sin t \cos t + \cos^2 t\right]$$

$$\times 2 \quad = 2e^{2t}$$

$$A = 2\pi \int_0^{\pi/2} e^t \sin t (\sqrt{2}) e^t \, dt = 2\sqrt{2}\pi \int_0^{\pi/2} e^{2t} \sin t \, dt \quad \times 1$$

$$= \frac{2\sqrt{2}\pi}{5}(2e^{\pi} + 1) \approx \underline{84.026}$$

Integrate by parts twice:

$$\int e^{2t}\sin t\,dt = \tfrac{1}{2}e^{2t}\sin t - \tfrac{1}{2}\int e^{2t}\cos t\,dt$$

$$\int e^{2t}\cos t\,dt = \tfrac{1}{2}e^{2t}\cos t + \tfrac{1}{2}\int e^{2t}\sin t\,dt$$

So $\int e^{2t}\sin t\,dt = \frac{2}{5}e^{2t}\left[\sin t - \frac{1}{2}\cos t\right] + C$

Whence: $\int_0^{\pi/2} e^{2t}\sin t\,dt = \frac{2}{5}e^{\pi} + \frac{1}{5}$.

MA 114H — Calculus II Fall 2014
Sections 009–010

Final Exam: 6-8 PM BS 116 December 16, 2014

Name: Solutions

Section: ________

Last 4 digits of student ID #: ______

- No books or notes may be used.
- Turn off all your electronic devices and do not wear ear-plugs during the exam.
- You may use a calculator, but not one which has symbolic manipulation capabilities or a QWERTY keyboard.
- Additional blank sheets for scratch work are available upon request.
- **Multiple Choice Questions:** Record your answers on the right of this cover page by marking the box corresponding to the correct answer.
- **Free Response Questions:** Show all your work on the page of the problem. Clearly indicate your answer and the reasoning used to arrive at that answer.

Unsupported answers for the free response questions might not receive credit!

Multiple Choice Answers

Question					
1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E

Exam Scores

Question	Score	Total
MC		20
5		16
6		16
7		16
8		16
9		16
Total		100

$\frac{1}{2}$

Record the correct answer to the following problems on the front page of this exam.

1. (5 points) The most general solution to the differential equation $xy' = y + x$ is:

A. $y(x) = x \ln x$

B. $y(x) = x^2 + Cx$, for any constant C

C. $y(x) = x \ln x + Cx$; for any constant C

D. $y(x) = x \ln x + x$

E. $y(x) = \ln x + Cx$, for any constant C

$y' - \frac{1}{x} y = 1$

$\mu = e^{\int p(x)dx} = e^{-\ln x} = \frac{1}{x}$

$\mu y = \int 1 \cdot \frac{1}{x} dx = \ln x + C$

$\rightarrow y(x) = x \ln x + Cx$

Check:

$y'(x) = \ln x + 1 + C$

$xy' = x\ln x + x + Cx$

$= y + x$ ✓

2. (5 points) The curve $c(t) = (x(t), y(t))$ with $x(t) = t\cos t$ and $y(t) = t \sin t$ is a spiral. The speed $v(t)$ at any time t is given by:

A. $v(t) = 1$

B. $v(t) = \sqrt{1+t^2}$

C. $v(t) = |t|$

D. $v(t) = \cos t - t \sin t$

E. $v(t) = \sin t + t\cos t$

$x'(t) = \cos t - t\sin t$

$y'(t) = \sin t + t\cos t$

$x'(t)^2 + y'(t)^2$

$= \cos^2 t - 2t \sin t \cos t + t^2 \sin^2 t$

$+ \sin^2 t + 2t \sin t \cos t + t^2 \cos^2 t$

$= 1 + t^2$

$v(t) = \sqrt{x'(t)^2 + y'(t)^2}$

$= \sqrt{1+t^2}$

Free Response Questions: Show your work!

8. (16 points) Consider the logistic equation

$$y'(x) = 5y(x)(y(x) - 2).$$

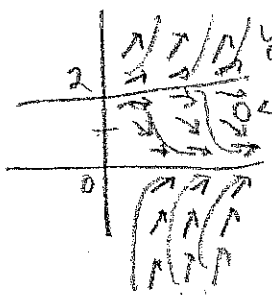
+4 (a.) Sketch the slope field for this equation and draw some solution curves.

+4 (b.) What are the equilibrium solutions?

+4 (c.) Find the most general solution to this differential equation.

+4 (d.) Use part (c.) to find the unique solution with $y(0) = 1$. What is the limit $\lim_{x\to\infty} y(x)$ for this solution?

a.) $f(y) = 5y(y-2)$ $f(0) = 0$ $f(2) = 2$.



$y > 2$ $f(y) > 0$

$0 < y < 2$ $f(y) < 0$

$y < 0$ $f(y) > 0$

b.) equilibrium sol'ns. $y = 0$ & $y = 2$.

c.) $\frac{dy}{y(y-2)} = 5dx$ or $\left(\frac{1}{y} - \frac{1}{y-2}\right)\left(-\frac{1}{2}\right) = \frac{1}{y(y-2)}$

$\frac{1}{2}[\log(y-2) - \log y] = \frac{1}{2}\log\left[\frac{y-2}{y}\right] = 5x + C$

$\frac{y-2}{y} = Ce^{10x}$ or $y(1 - Ce^{10x}) = 2$

$$\boxed{y(x) = \frac{2}{1 - Ce^{10x}}}$$

d) $y(0) = \frac{2}{1-C} = 1$

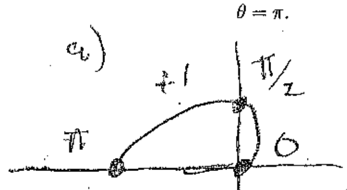
$C = -1$ $y(x) = \frac{2}{1+e^{10x}}$ and $\lim_{x\to\infty} y(x) = 0$.

Free Response Questions: Show your work!

7. (16 points total) Sketch the polar curve $r(\theta) = 1 - \cos\theta$ for $\theta \in [0, \pi]$.

(a.) (8 points) Compute the area bounded by this curve and the rays at $\theta = 0$ and $\theta = \pi$.

a)



θ	$r(\theta)$
0	0
$\frac{\pi}{2}$	1
π	2

π +2

$$A = \frac{1}{2}\int_0^{\pi} r^2(\theta)\,d\theta = \frac{1}{2}\int_0^{\pi} (1-\cos\theta)^2\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi} (1 - 2\cos\theta + \cos^2\theta)\,d\theta$$

(b.) (8 points) Compute the arc length of the curve for $\theta \in [0, \pi]$. You might need: $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$ and $\cos^2 t = \frac{1}{2}(1 + \cos 2t)$.

$$= \frac{1}{2}\int_0^{\pi} \left[\tfrac{3}{2} - 2\cos\theta + \tfrac{1}{2}\cos 2\theta\right] d\theta \quad +2$$

+1

$$= \frac{1}{2}\left[\frac{3}{2}\pi - 2\sin\theta\Big|_0^{\pi} + \frac{1}{4}\sin 2\theta\Big|_0^{\pi}\right] = \frac{3\pi}{4}$$

+2

$$b)\ L = \int_0^{\pi} \left[r(\theta)^2 + r'(\theta)^2\right]^{\frac{1}{2}} d\theta \quad +2$$

$$r'(\theta) = \sin\theta$$

$$r(\theta)^2 + r'(\theta)^2 = 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta$$

$$= 2(1 - \cos\theta)$$

$$+2 \quad = 4\sin^2(\theta/2)$$

$$L = \int_0^{\pi} 2\sin(\theta/2)\,d\theta = -4\cos\left(\frac{\theta}{2}\right)\Big|_0^{\pi}$$

$$= 4. \quad +4$$

Record the correct answer to the following problems on the front page of this exam.

3. (5 points) The radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} x^n$ is

A. $R = \infty$

B. $R = 1$

C. $R = 2$

D. $R = \frac{1}{2}$

E. $R = 0$

Ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n}\right|$

$$= |x| \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = |x|$$

Need $|x| < 1$ so $R = 1$.

4. (5 points) The first three nonzero terms of the Taylor expansion of $g(x) = e^x \cos x$ about $x = 0$ are:

A. $1 - x^2 + x^4$

B. $1 - \frac{x^2}{2} + \frac{x^3}{3!}$

C. $x + \frac{x^2}{2} + \frac{x^4}{4!}$

D. $1 + 2x + \frac{x^2}{2}$

E. $1 + x - \frac{x^3}{2}$

Multiply

$$e^x = 1 + x + \frac{x^2}{2} + $$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots$$

$$e^x \cos x = \left(1 + x + x^2/2 + \ldots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} \ldots\right)$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{4!} + x - \frac{x^3}{2} + \frac{x^2}{2} \cdots$$

$$= 1 + x - \frac{x^3}{2}$$

5. (16 points)

(a.) Compute the indefinite integral:

$$\int \sqrt{\cosh x - 1}\, dx.$$

+3

Hint: Recall the identity $\cosh^2 x - \sinh^2 x = 1$.

$u = \cosh x$ +1, $u^2 - 1 = \sinh^2 x$

$du = \sinh x\, dx$ +1

$$\int [u-1]^{\frac{1}{2}} \frac{du}{\sinh x} = \int \frac{[u-1]^{\frac{1}{2}}}{[u^2-1]^{\frac{1}{2}}} du = \int \left[\frac{u-1}{(u-1)(u+1)}\right]^{\frac{1}{2}} du$$ +2

$$= \int \frac{1}{\sqrt{u+1}} du = 2[u+1]^{\frac{1}{2}} + C = 2[\cosh x + 1]^{\frac{1}{2}} + C$$ +1, +2, +1

(b.) Compute the following indefinite integral:

$$J = \int \frac{dx}{x^2\sqrt{x^2-1}}.$$

$x = \sec\theta$ +1

$dx = \sec\theta \tan\theta\, d\theta$ +1

$\tan^2\theta + 1 = \sec^2\theta$

$$J = \int \frac{\sec\theta \tan\theta}{\sec^2\theta \sqrt{\sec^2\theta - 1}} d\theta = \int \cos\theta\, d\theta = \sin\theta + C$$ +3, +1

$\cos\theta = \frac{1}{x}$

$1 - \cos^2\theta = \sin^2\theta = \frac{x^2-1}{x^2}$

$\sin\theta = \sqrt{\frac{x^2-1}{x^2}}$ +2

$J = \frac{\sqrt{x^2-1}}{x} + C$

6. (16 points) Consider the power series:

$$\sum_{n=0}^{\infty} \frac{(-1)^n n}{2^n}(x+1)^n.$$

(a.) (8 points) Compute the radius of convergence of the series.

$$+2\ \lim_{n\to\infty} \left| \frac{(n+1)(x+1)^{n+1}}{2^{n+1}} \frac{2^n}{n(1+x)^n} \right| = |x+1| \lim_{n\to\infty} \left(\frac{n+1}{n}\right)\frac{1}{2}$$ +2

+1 $= \frac{1}{2}|x+1| < 1$

+1 $|x+1| < 2$

+2 $R = 2$ and we have absolute conv. in $(-3, 1)$

$-3 \quad -1 \quad 0 \quad 1$

(b.) (8 points) Compute the interval of convergence. This means check the behavior of the series at the end points.

Check endpts:

$x = -3$ +1: $\sum_{n=0}^{\infty} \frac{(-1)^n n}{2^n}(-2)^n = \sum_{n=0}^{\infty} n$ diverges +1 +1 +1

since $a_n = n$ does not go to 0

$x = 1$ +1: $\sum_{n=0}^{\infty} (-1)^n n$ +2 diverges for the same reason +1

Interval of Conv. $(-3, 1)$ (abs. conv.) and diverges everywhere else.