

Name: Solution

Section: _____

Last 4 digits of student ID #: _____

- No books or notes may be used.
- Turn off all your electronic devices and do not wear ear-plugs during the exam.
- You may use a calculator, but not one which has symbolic manipulation capabilities or a QWERTY keyboard.
- Additional blank sheets for scratch work are available upon request.
- **Multiple Choice Questions:**
Record your answers on the right of this cover page by marking the box corresponding to the correct answer.
- **Free Response Questions:**
Show all your work on the page of the problem. Clearly indicate your answer and the reasoning used to arrive at that answer.

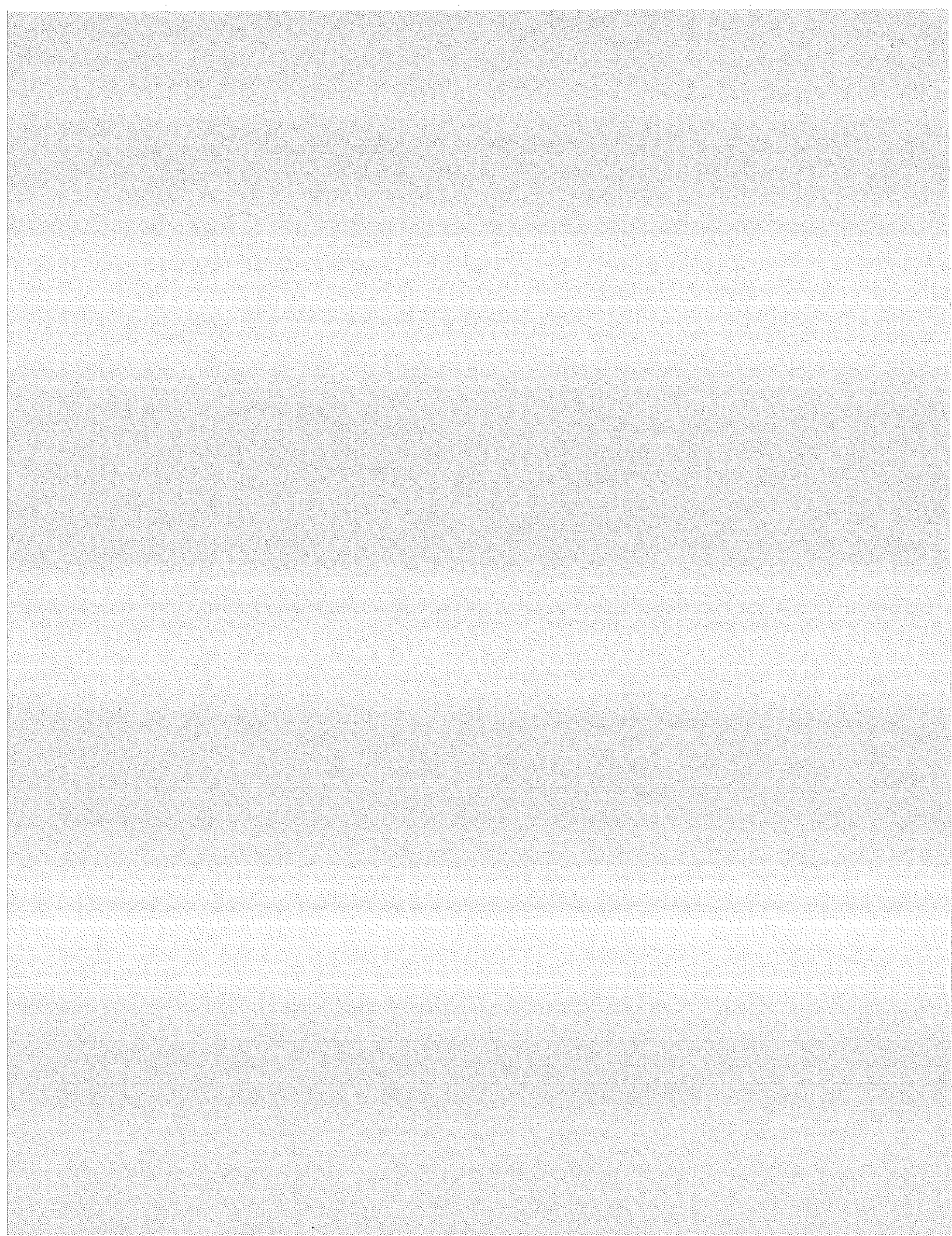
Multiple Choice Answers

Question					
1	A	B	<input checked="" type="radio"/>	D	E
2	A	<input checked="" type="radio"/>	C	D	E
3	A	<input checked="" type="radio"/>	C	D	E
4	A	B	C	<input checked="" type="radio"/>	E

Exam Scores

Question	Score	Total
MC		20
5		16
6		16
7		16
8		16
9		16
Total		100

Unsupported answers for the free response questions might not receive credit!



Record the correct answer to the following problems on the front page of this exam.

1. (5 points) The definite integral

$$\int_0^{\pi/4} \tan x \sec^2 x \, dx$$

is equal to:

A. $\frac{1}{4}$

B. $\frac{\pi}{2}$

C. $\frac{1}{2}$

D. $\frac{\pi}{4}$

E. 1

$$\frac{d \tan x}{dx} = \sec^2 x$$

$$\text{let } u = \tan x$$

Substitute

$$\int_{x=0}^{\pi/4} u \, du = \frac{1}{2} \tan^2 x \Big|_0^{\pi/4} = \frac{1}{2}$$

2. (5 points) Consider the region in the first quadrant bounded by the curve $y(x) = x^2$ and the line $x = 2$. The volume of the solid of revolution obtain by rotating this region around the x -axis is:

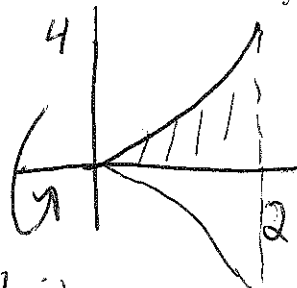
A. $\frac{30\pi}{5}$

B. $\frac{32\pi}{5}$

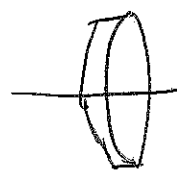
C. $\frac{32\pi}{3}$

D. $\frac{16\pi}{5}$

E. None of the above.



disk



$$\pi y^2 = \pi x^4$$

$$\int_0^2 \pi x^4 \, dx = \frac{\pi x^5}{5} = \frac{32\pi}{5}$$

Record the correct answer to the following problems on the front page of this exam.

3. (5 points) The indefinite integral $\int x \cos x \, dx$ is equal to which one of the following:

A. $x(\sin x + \cos x) + C$

B. $x \sin x + \cos x + C$

C. $x \cos x + \sin x + C$

D. $\sin x + \cos x + C$

E. None of the above.

Integrate by parts

$$u = x \quad v' = \cos x$$

$$u' = 1 \quad v = \sin x$$

$$\int uv' = uv - \int v u'$$

$$= x \sin x - \int \sin x + C$$

$$= x \sin x + \cos x + C$$

Check $\frac{d}{dx} (x \sin x + \cos x + C) = \cancel{\sin x} + x \cos x - \cancel{\sin x}$
 $= x \cos x \checkmark$

4. (5 points) The average of the function $g(x) = \tan x$ over the interval $[0, 2]$ is equal to

A. 0

B. $-\frac{1}{2} \ln 2$

C. $\ln(|\cos 2|)$

D. $-\frac{1}{2} \ln(|\cos 2|)$

E. $\frac{1}{2} |\cos 2|$

$\frac{1}{b-a} \int_a^b g(x) \, dx$ is the average.

$$\frac{1}{2} \int_0^2 \tan x \, dx$$

$$= \frac{1}{2} \int_0^2 \frac{\sin x}{\cos x} \, dx = -\frac{1}{2} \int_{\cos 2}^1 \frac{1}{u} \, du$$

$$u = \cos x$$

$$du = -\sin x \, dx$$

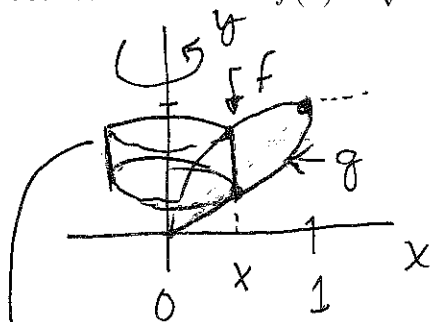
$$= -\frac{1}{2} \ln |u| \Big|_1^{\cos 2}$$

$$= -\frac{1}{2} \ln |\cos 2|$$

NOTE This is a bad problem since $\pi \in [0, 2]$ and $\lim_{x \rightarrow \pi} \tan x = \infty$ so one can't integrate over $[0, 2]$. We graded it as written here, though.

Free Response Questions: Show your work!

5. (16 points) Find the volume of the solid of revolution formed by rotating the region between the curves $f(x) = \sqrt{x}$ and $g(x) = x^2$ around the y -axis.



$$f(x) = g(x)$$

$$\sqrt{x} = x^2 \quad x=0$$

$$\text{and } x = x^4$$

$$x=1$$

rotate around $x=0$.

Shell method: $2\pi x (f(x) - g(x)) \Delta x = \Delta V$

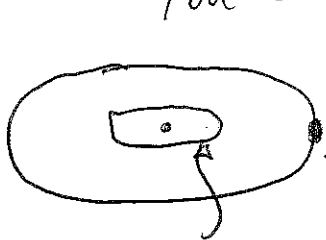
$$0 \leq x \leq 1$$

$$V = \int_0^1 2\pi x (x^{1/2} - x^2) dx = 2\pi \int_0^1 (x^{3/2} - x^3) dx$$

$$= 2\pi \left[\frac{2x^{5/2}}{5} - \frac{x^4}{4} \right]_0^1 = 2\pi \left(\frac{2}{5} - \frac{1}{4} \right)$$

$$= 2\pi \left(\frac{8-5}{20} \right) = \boxed{\frac{3\pi}{10}}$$

You could also use the washer method.



$$\Delta V = \pi (R_{out}^2 - R_{in}^2) \quad 0 \leq y \leq 1$$

$$R_{out}^2(y) = y$$

$$R_{in}^2(y) = y^4$$

$$V = \pi \int_0^1 (y - y^4) dy = \pi \left(\frac{y^2}{2} - \frac{y^5}{5} \right) \Big|_0^1$$

$$= \frac{3\pi}{10}$$

$$f$$

$$y = x^{1/2}$$

$$x = y^2$$

$$R_{in}$$

$$g$$

$$y = x^2$$

$$x = y^{1/2}$$

$$R_{out}$$

Free Response Questions: Show your work!

6. (16 points) Taylor's Theorem

(a.) (10 points) Find the Taylor polynomial $T_2(x)$ for the function $f(x) = \sin(2x)$ about $a = \frac{\pi}{2}$.

(b.) (6 points) Using the formula

$$R_n(x) = \frac{1}{n!} \int_{\frac{\pi}{2}}^x (x-u)^n f^{(n+1)}(u) du$$

estimate the error $|f(x) - T_2(x)|$ for $x \in [\frac{\pi}{4}, \frac{3\pi}{4}]$.

$$(a.) T_n(x) = \sum_{j=0}^n \frac{f^{(j)}(a)}{j!} (x-a)^j$$

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

$$f(x) = \sin 2x$$

$$f(\frac{\pi}{2}) = 0$$

$$f'(x) = 2 \cos 2x$$

$$f'(\frac{\pi}{2}) = -2$$

$$f''(x) = -4 \sin 2x$$

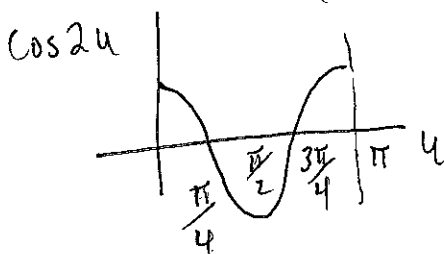
$$f''(\frac{\pi}{2}) = 0$$

$$f'''(x) = -8 \cos 2x \quad (\text{needed for } R_2(x))$$

$$T_2(x) = -2(x - \frac{\pi}{2})$$

$$(b) R_2(x) = \frac{1}{2} \int_{\frac{\pi}{2}}^x (x-u)^2 f'''(u) du = \frac{1}{2} \int_{\frac{\pi}{2}}^x (x-u)^2 (-8 \cos 2u) du$$

$$|R_2(x)| \leq \frac{1}{2} \cdot 8 \left(\max_{u \in [\frac{\pi}{4}, \frac{3\pi}{4}]} |\cos 2u| \right) \frac{|x - \frac{\pi}{2}|^3}{3} = \frac{4}{3} |x - \frac{\pi}{2}|^3$$



$\max_{u \in [\frac{\pi}{4}, \frac{3\pi}{4}]} |\cos(2u)| = 1$ and \max

$$|R_2(x)| \leq \frac{4}{3} |x - \frac{\pi}{2}|^3 \text{ and } |R_2(x)| \leq \frac{\pi^3}{3 \cdot 4^2} \text{ for } x \in [\frac{\pi}{4}, \frac{3\pi}{4}]$$

Free Response Questions: Show your work!

7. (16 points total) Compute the following integrals

(a.) (8 points)

Integrate by parts $J = \int e^{2x} \cos x \, dx = uv - \int v u' = \frac{1}{2} e^{2x} \cos x + \frac{1}{2} \int e^{2x} \sin x$

$u = \cos x$ $v' = e^{2x}$
 $u' = -\sin x$ $v = \frac{1}{2} e^{2x}$

Repeat:

$\int e^{2x} \sin x = \frac{1}{2} e^{2x} \sin x - \frac{1}{2} \int e^{2x} \cos x \, dx$

$J = \frac{1}{2} e^{2x} \cos x + \frac{1}{2} \left(\frac{1}{2} e^{2x} \sin x - \frac{1}{2} \int e^{2x} \cos x \, dx \right)$ $v' = e^{2x}$ $u = \sin x$
 $v = \frac{1}{2} e^{2x}$ $u' = \cos x$

$= \frac{1}{2} e^{2x} \cos x + \frac{1}{4} e^{2x} \sin x - \frac{1}{4} \int e^{2x} \cos x \, dx$

Move the integral left:

$\frac{5}{4} \int e^{2x} \cos x \, dx = \frac{1}{2} e^{2x} \cos x + \frac{1}{4} e^{2x} \sin x + C$

so $J = \frac{2}{5} e^{2x} \cos x + \frac{1}{5} e^{2x} \sin x + C$

(b.) (8 points)

$\int \sin^3 x \cos^2 x \, dx$

$\int \sin x (\sin^2 x \cos^2 x) \, dx = \int \sin x (1 - \cos^2 x) \cos^2 x \, dx$

$= \int \cos^2 x \sin x - \int \cos^4 x \sin x \, dx$

$\int \sin^3 x \cos^2 x = -\frac{1}{3} \cos^3 x + \frac{1}{5} \cos^5 x + C$

Check (b)

$\frac{d}{dx} \left(-\frac{1}{3} \cos^3 x + \frac{1}{5} \cos^5 x + C \right) = +\cos^2 x \sin x - \cos^4 x \sin x$ ✓
 $= (1 - \cos^2 x) \cos^2 x \sin x = \cos^2 x \sin^3 x$

Check (a) $\frac{d}{dx} \left(\frac{2}{5} e^{2x} \cos x + \frac{1}{5} e^{2x} \sin x + C \right) = \frac{4}{5} e^{2x} \cos x - \frac{2}{5} e^{2x} \sin x$
 $+ \frac{2}{5} e^{2x} \sin x + \frac{1}{5} e^{2x} \cos x = e^{2x} \cos x$ ✓

Free Response Questions: Show your work!

8. (16 points)

(a.) (8 points) Find the Taylor series of the function

$$f(x) = \frac{1}{(2-3x)}$$

about the origin. What is the radius of convergence?

$$f(x) = \frac{1}{2} \frac{1}{1 - \frac{3}{2}x} = \frac{1}{2} \sum_{j=0}^{\infty} \left(\frac{3}{2}x\right)^j \quad R = \frac{2}{3} \text{ since we need } \left|\frac{3}{2}x\right| < 1.$$

This is a geometric series

$$\frac{1}{2-3x} = \frac{1}{2} \sum_{j=0}^{\infty} \left(\frac{3}{2}x\right)^j$$

(b.) (8 points) Use the result from part (a.) to find the Taylor series about the origin of the function

$$g(x) = \frac{1}{(2-3x)^2}$$

Use differentiation term by term

$$\frac{d}{dx} \frac{1}{2-3x} = \left(\frac{-1}{(2-3x)^2}\right) \cdot (-3) = 3 \frac{1}{(2-3x)^2}$$

$$\text{so } \frac{1}{(2-3x)^2} = \frac{1}{3} \frac{d}{dx} \left(\frac{1}{2-3x}\right) = \frac{1}{6} \frac{d}{dx} \left(\frac{1}{1 - \frac{3}{2}x}\right) = \frac{1}{6} \sum_{j=0}^{\infty} j \left(\frac{3}{2}x\right)^{j-1} \left(\frac{3}{2}\right)$$

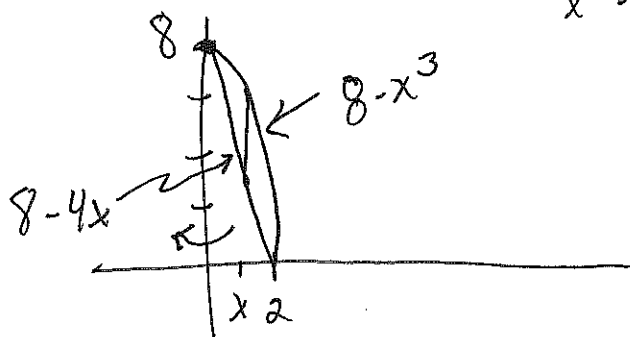
$$\text{so } \frac{1}{(2-3x)^2} = \frac{1}{4} \sum_{j=1}^{\infty} j \left(\frac{3}{2}x\right)^{j-1}$$

Free Response Questions: Show your work!

9. (16 points total)

(a.) (2 points) Sketch the region in the (x, y) plane bounded by the curves $y(x) = 8 - x^3$ and $y(x) = 8 - 4x$, for $x \geq 0$.

Intersection pts: $8 - x^3 = 8 - 4x$
 $x^3 = 4x$ so $x = 0$ and $x^2 = 4$ $x = 2$ ($x \geq 0$)



(b.) (14 points) Find the volume of the solid of revolution obtained by rotating the region described in (a.) about the y -axis.

Shell Method:

$$V = \int_0^2 2\pi x ((8 - x^3) - (8 - 4x)) dx = \int_0^2 2\pi x (4x - x^3) dx$$

$$= 2\pi \left(\int_0^2 4x^2 dx - \int_0^2 x^4 dx \right)$$

$$= 2\pi \left(\frac{4x^3}{3} \Big|_0^2 - \frac{x^5}{5} \Big|_0^2 \right) = 2\pi \left(\frac{32}{3} - \frac{32}{5} \right) = 2\pi \left(\frac{160 - 96}{15} \right)$$

$$V = \frac{128\pi}{15}$$

