MA 114H — Calculus II Fall 2014
Sections 009–010

Exam 3 November 18, 2014

Name: Solutions

Section: ______

Last 4 digits of student ID #: ______

- No books or notes may be used.
- Turn off all your electronic devices and do not wear ear-plugs during the exam.
- You may use a calculator, but not one which has symbolic manipulation capabilities or a QWERTY keyboard.
- Additional blank sheets for scratch work are available upon request.
- **Multiple Choice Questions:** Record your answers on the right of this cover page by marking the box corresponding to the correct answer.
- **Free Response Questions:** Show all your work on the page of the problem. Clearly indicate your answer and the reasoning used to arrive at that answer.

Multiple Choice Answers

Question					
1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E

Exam Scores

Question	Score	Total
MC		20
5		16
6		16
7		16
8		16
9		16
Total		100

Unsupported answers for the free response questions might not receive credit!

Record the correct answer to the following problems on the front page of this exam.

1. (5 points) The region lying under the graph of $f(x) = x^3$ and over the interval $x \in [0, 2]$ has centroid (that is, the center of mass coordinates):

A. $(\frac{8}{5}, 2)$

B. $(\frac{8}{5}, \frac{16}{7})$

C. $(1, 4)$

D. $(\frac{8}{3}, \frac{16}{7})$

E. $(1, \frac{11}{4})$

$X_{CM} = \frac{M_y}{M} = \frac{8}{5}$

$M_y = \int_0^2 x f(x)\,dx = \int_0^2 x^4\,dx = 32/5$

$Y_{CM} = \frac{M_x}{M}$

$M_x = \frac{1}{2}\int_0^2 f(x)^2\,dx = \frac{1}{2}\int_0^2 x^6\,dx = \frac{x^7}{2\cdot 7}\Big|_0^2 = \frac{128}{2\cdot 7} = \frac{64}{7}$

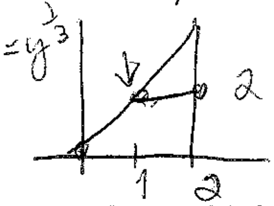
$M = \int_0^2 x^3\,dx = \frac{x^4}{4}\Big|_0^2 = \frac{16}{4} = 4$

so $Y_{CM} = \frac{64}{4\cdot 7} = \frac{16}{7}$

You can also compute

$M_x = \int_0^8 y(2 - y^{\frac{1}{3}})\,dy = 64 - \frac{3y^{7/3}}{7}\Big|_0^8 = 64 - \frac{3\cdot 128}{7}$

$Y_{CM} = 16 - 96/7 = 16/7$



2. (5 points) The surface area of the solid of revolution formed by rotation the graph of the function $y(x) = 2x + 1$ about the x-axis from $x = 1$ to $x = 2$ is:

A. $8\pi\sqrt{5}$

B. $\pi\sqrt{5}$

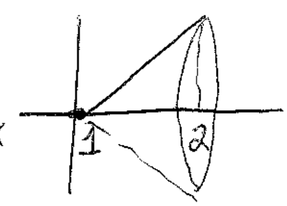
C. 8π

D. $4\pi\sqrt{5}$

E. 2π

$f'(x) = 2$

$S = 2\pi \int_1^2 f(x)\sqrt{1 + f'(x)^2}\,dx$



$= 2\pi\sqrt{5} \int_1^2 (2x+1)\,dx$

$= 2\pi\sqrt{5}\left(x^2\Big|_1^2 + x\Big|_1^2\right)$

$= 2\pi\sqrt{5}(3+1)$

$= 8\pi\sqrt{5}$

Record the correct answer to the following problems on the front page of this exam.

3. (5 points) The partial fraction decomposition of

$$\frac{1}{(x^2+2)(x-3)}$$

is equal to which one of the following:

A. $\frac{-1}{11}\left(\frac{x+3}{x^2+2}+\frac{1}{x-3}\right)$

B. $\frac{1}{11}\left(-\frac{x+3}{x^2+2}+\frac{1}{x-3}\right)$

C. $\frac{-1}{11}\left(\frac{x-3}{x^2+2}+\frac{1}{x-3}\right)$

D. $\left(\frac{3}{x^2+2}+\frac{1}{x-3}\right)$

E. $\frac{-1}{11}\left(\frac{x}{x^2+2}+\frac{1}{x-3}\right)$

x^2+2 irreducible

$= \frac{Ax+B}{x^2+2} + \frac{C}{x-3}$

$1 = (Ax+B)(x-3) + Cx^2+2C$

$= Ax^2+Bx-3Ax-3B+Cx^2+2C$

$= (A+C)x^2+(B-3A)x+(2C-3B)$

$\begin{cases} A+C=0 \\ B-3A=0 \\ 2C-3B=1 \end{cases}$

$A=-C$

$B=3A=-3C$

$2C-3(-3C)=1$

$11C=1$

$C=\frac{1}{11}$

$A=-\frac{1}{11}$

$B=\frac{-3}{11}$

4. (5 points) The derivative of the function $f(x)=\cosh(3x^2)$ is equal to:

A. $-6x\sin(3x^2)$

B. $-6x\sinh(3x^2)$

C. $3x^2\sinh(3x^2)$

D. $6x\sinh(3x^2)$

E. $6x\cosh(3x^2)$

$f'(x) = \sinh(3x^2)\cdot 6x$

$= 6x\sinh(3x^2)$

Free Response Questions: Show your work!

5. (16 points) Evaluate the following indefinite integral:

$$I = \int \frac{dx}{(x^2+1)^2}.$$

You might need:

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$(x^2+1)^2 = \sec^4\theta$

$$I = \int \frac{\sec^2\theta}{\sec^4\theta}\, d\theta = \int \cos^2\theta \, d\theta$$

$$= \frac{1}{2}\int d\theta + \frac{1}{2}\int \cos 2\theta \, d\theta = \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + C$$

$\theta = \tan^{-1} x$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\tan\theta\cos^2\theta = \frac{2x}{\sec^2\theta} = \frac{2x}{1+x^2}$$

$$\boxed{\Rightarrow \quad I = \frac{1}{2}\tan^{-1}x + \frac{1}{2}\frac{x}{1+x^2} + C}$$

Check $I' = \frac{1}{2}\frac{1}{1+x^2} + \frac{1}{2}\frac{1}{1+x^2} - \frac{1}{2}\frac{x \cdot 2x}{(1+x^2)^2}$

$$= \frac{1+x^2}{(1+x^2)^2} - \frac{x^2}{(1+x^2)^2} = \frac{1}{(1+x^2)^2} \quad \checkmark$$

Free Response Questions: Show your work!

6. (16 points) Compute the arc length of the curve $f(x) = e^x$ on the interval $[0, a]$, for any $a > 0$. Use the substitution $u = \sqrt{1+e^{2x}}$, note that $e^{2x} = u^2 - 1$, and use partial fractions to do the u integral.

$$L = \int_0^a \sqrt{1+f'(x)^2}\,dx = \int_0^a [1+e^{2x}]^{\frac{1}{2}}\,dx$$

$$u = [1+e^{2x}]^{\frac{1}{2}}$$

$$du = \frac{1}{2}[1+e^{2x}]^{-\frac{1}{2}}\,2e^{2x}\,dx = \frac{e^{2x}}{[1+e^{2x}]^{\frac{1}{2}}}\,dx = \frac{u^2-1}{u}\,dx$$

$$\text{so} \quad dx = \frac{u}{u^2-1}\,du$$

$$L = \int_{\sqrt{2}}^{[1+e^{2a}]^{\frac{1}{2}}} \frac{u^2}{u^2-1}\,du$$

Indefinite Integral: $\displaystyle\int \frac{u^2}{u^2-1}\,du = \int\left[\frac{u^2-1}{u^2-1} + \frac{1}{u^2-1}\right]du$

$$= u + \int \frac{du}{u^2-1}$$

2nd indef. integral:

$$\frac{1}{u^2-1} = \frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}$$

$$1 = A(u+1) + B(u-1) = (A+B)u + (A-B)$$

$$A = -B$$

$$A - B = 1 \quad A = \frac{1}{2} \quad B = -\frac{1}{2}$$

$$\int \frac{du}{u^2-1} = \frac{1}{2}\left(\ln|u-1| - \ln|u+1|\right) = \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|$$

(in our definite integral $u \geq \sqrt{2} > 1$)

$$L = \int_{\sqrt{2}}^{[1+e^{2a}]^{\frac{1}{2}}} \frac{u^2}{u^2-1}\,du = \left\{[1+e^{2a}]^{\frac{1}{2}} - \sqrt{2}\right\} + \frac{1}{2}\left\{\ln\left(\frac{\sqrt{1+e^{2a}}-1}{\sqrt{1+e^{2a}}+1}\right) - \ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right\}.$$

Free Response Questions: Show your work!

7. (16 points total) Differential equations.

(a.) (8 points) Find the most general solution:

$$y'(x) = 2(y(x) - 3)$$

Separate variables

$\frac{dy}{y-3} = 2dx$ $\quad \ln|y-3| = 2x + C$

$\boxed{y = 3 + Ae^{2x}}$

$y'(x) = 2Ae^{2x} = 2(y-3)$ ✓

(b.) (8 points) Find the unique positive solution for $x \geq 0$ to the initial-value problem:

$$y(x)y'(x) = x^2 e^{-2y^2(x)}, \quad y(0) = 0,$$

with $x \geq 0$ and $y(x) \geq 0$.

$yy' = x^2 e^{-2y^2}$ Separate Variables $(ye^{2y^2})dy = x^2 dx$

$\int ye^{2y^2} dy = \int \frac{1}{4} e^{z} dz = \frac{1}{4} e^{2y^2}$

$z = 2y^2$
$dz = 4y\,dy$

So $\frac{1}{4} e^{2y^2} = \frac{1}{3}x^3 + C$ $\quad x > 0, y > 0$

$2y^2 \quad \ln\left[\frac{4}{3}x^3 + C\right]$

General: $y(x) = \left[\frac{1}{2}\ln\left[\frac{4}{3}x^3 + C\right]\right]^{\frac{1}{2}}$

Initial Condition $y(0) = 0$ so $\frac{1}{2}\ln C = 0 \Rightarrow C = 1$

Soln. $\boxed{y(x) = \left\{\frac{1}{2}\ln\left(\frac{4}{3}x^3 + 1\right)\right\}^{\frac{1}{2}}}$

Check:
$y' = \frac{1}{2}\left(\frac{1}{2}\ln\left(\frac{4}{3}x^3 + C\right)\right)^{-\frac{1}{2}}$
$\times \frac{1}{2} \frac{4x^2}{\frac{4}{3}x^3 + C}$
or
$yy' = \left(\frac{x^2}{\frac{4}{3}x^3 + C}\right)$
$= x^2 e^{-2y^2}$ ✓

Free Response Questions: Show your work!

8. (16 points) Newton's Law of Cooling states that the rate of change of the temperature $T(t)$ at time $t \geq 0$, measured in hours, of a body initially at temperature T_1 in an ambient environment at temperature $T_0 < T_1$ satisfies the initial value problem:

$$T'(t) = -k(T - T_0), \quad T(t=0) = T_1, \quad k > 0.$$

How long does it take a cup of hot chocolate initially at temperature $T_1 = 80^oC$ in an room at 20^oC to cool to 50^oC if $k = \ln 2/hour$.

$T_0 = 20 \quad T_1 = 80$

$T'(t) = -k(T - T_0)$ or $\frac{dT}{T - T_0} = -k\,dt$

$\ln(T - T_0) = -kt$

$T = T_0 + Ae^{-kt}$

$T(t) = 20 + Ae^{-kt}$

Initial Condition: $T(0) = 80 = 20 + A$

$A = 60$.

$T(t) = 20 + 60e^{-kt} = 20 + 60e^{-(\ln 2)t}$

At what time t_1 is $T(t_1) = 50$?

$20 + 60e^{-(\ln 2)t_1} = 50$

$e^{-(\ln 2)t_1} = 1/2$

$-(\ln 2)t_1 = -\ln 2$

$\boxed{t_1 = 1 \text{ hour}}$

Free Response Questions: Show your work!

9. (16 points total) Let's consider the Simpson approximation method applied to the following integral.

(a.) (6 points) Write the integral for the arc length of the curve $f(x) = \sin x$ with $x \in [0, \pi]$. Do not evaluate the integral.

(b.) (6 points) Write Simpson's approximation for $N = 4$ for the integral found in part (a.) above.

(c.) (4 points) Evaluate Simpson's approximation of part (b.) exactly. This means leave quantities like $\sqrt{2}$. Do not use a calculator.

a) $f'(x) = \cos x$

$$L = \int_0^{\pi} \sqrt{1+\cos^2 x}\, dx \qquad \text{let } g(x) = \sqrt{1+\cos^2 x}$$

b) Partition $[0, \pi]$ into 4 subintervals $\Delta x = \pi/4$

$0 \quad \pi/4 \quad \pi/2 \quad 3\pi/4 \quad \pi$

$$S = \frac{\Delta x}{3}\left[g(a) + 4g(x_1) + 2g(x_2) + 4g(x_3) + g(b)\right]$$

$$= \frac{\pi}{12}\left[g(0) + 4g(\pi/4) + 2g(\pi/2) + 4g(3\pi/4) + g(\pi)\right]$$

where $g(x) = \sqrt{1+\cos^2 x}$

c.) Calculate: $g(0) = \sqrt{2}$ $\quad g(\pi/4) = \sqrt{3/2}$ $\quad g(\pi/2) = 1$

$g(3\pi/4) = \sqrt{3/2}$ $\quad g(\pi) = \sqrt{2}$

$$S = \frac{\pi}{12}\left[2\sqrt{2} + 4\sqrt{6} + 2\right] = \frac{\pi}{6}\left[\sqrt{2} + 2\sqrt{6} + 1\right]$$