Solution

WA 2-1

WA 2 Error in the Trapezoidal Rule

(a) $n = 1$ $\quad T_1(f) = \left(\frac{b-a}{2}\right)\left(f(a) + f(b)\right)$ (1)

$$R_1(f) = T_1(f) - I(f)$$
$$= \left(\frac{b-a}{2}\right)\left(f(a) + f(b)\right) - \int_a^b f(x)\,dx \quad (2)$$

(b) Let $R_1(f) = \int_a^b g(x) f'(x)\,dx$. We want to find g so this equals (2). Integrate by parts with: $\quad v'(x) = f'(x) \quad u(x) = g(x)$

$v(x) = f(x) \quad u'(x) = g'(x)$

then:

$$\int_a^b g(x) f'(x)\,dx = g(b)f(b) - g(a)f(a) - \int_a^b g'(x) f(x)\,dx \quad (3)$$

Comparing with (2) we see $g'(x) = 1$ so $\boxed{g(x) = x + C}$ for a constant C. To fix C compare the first terms of (2) and (3):

$$(b+c)f(b) - (a+C)f(a) = \left(\frac{b-a}{2}\right)\left(f(a) + f(b)\right).$$

We find:

$$b + C = \left(\frac{b-a}{2}\right) \quad \text{and} \quad -(a+C) = \frac{b-a}{2}$$

or

$$b + C = -(a+C) \implies \boxed{C = -\left(\frac{a+b}{2}\right)}$$

Let $x_M = \frac{a+b}{2}$. We get:

$$\boxed{R_1(f) = \int_a^b (x - x_M) f'(x)\,dx} \quad (4)$$

WA2-2

(c) Integrate by parts in (4) with

$$u(x) = f'(x) \qquad v'(x) = (x - x_M)$$
$$u'(x) = f''(x) \qquad v(x) = \tfrac{1}{2}(x - x_M)^2$$

$$R_1(f) = f'(x)\,\tfrac{1}{2}(x - x_M)^2 \Big|_a^b - \int_a^b \frac{1}{2}(x - x_M)^2 f''(x)\,dx \qquad (5)$$

Evaluate the 1st term:

$$(b - x_M)^2 = \left(b - \left(\frac{a+b}{2}\right)\right)^2 = \left(\frac{b-a}{2}\right)^2$$

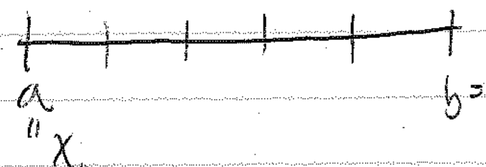
$$(a - x_M)^2 = \left(\frac{b-a}{2}\right)^2$$

$$f'(x)\,\tfrac{1}{2}(x - x_M)^2 \Big|_a^b = \big(f'(b) - f'(a)\big)\left(\frac{b-a}{2}\right)^2$$

$$= \left(\frac{b-a}{2}\right)^2 \int_a^b f''(x)\,dx \qquad (6)$$

Using (6) in (5) gives:

$$R_1(f) = \frac{1}{2}\int_a^b \left\{ \left(\frac{b-a}{2}\right)^2 - (x - x_M)^2 \right\} f''(x)\,dx \qquad (7)$$

(d) 

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

Equal length: $\Delta x_j = x_j - x_{j-1} = \dfrac{b-a}{n}$.

$$\int_a^b f(x)\,dx = \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(x)\,dx \qquad (8a)$$

WA 2-3

Write $T_n(f)$ as follows. $\Delta x = \left(\frac{b-a}{n}\right)$

$$T_n(f) = \frac{\Delta x}{2}\left[f(a) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(b)\right]$$

$$= \frac{\Delta x}{2}\left[f(x_0) + f(x_1)\right] + \frac{\Delta x}{2}\left[f(x_1)+f(x_2)\right] + \cdots$$

$$+ \frac{\Delta x}{2}\left[f(x_{n-1}) + f(x_n)\right]$$

$$= \sum_{j=1}^{n} \underbrace{\frac{\Delta x}{2}\left[f(x_{j-1}) + f(x_j)\right]}_{\text{this is } T_1(f \text{ on interval } [x_{j-1}, x_j])} \qquad (8b)$$

Apply the results of parts (a)-(c) to this with $[a,b]$ there replaced by $[x_{j-1}, x_j]$:

$$R_1(f\,|\,[x_{j-1},x_j]) = -\int_{x_{j-1}}^{x_j} f(x)\,dx + T_1(f\,|\,[x_{j-1},x_j])$$

$$= \frac{1}{2}\int_{x_{j-1}}^{x_j} \left\{ \left(\frac{b-a}{2n}\right)^2 - (x-x_m)^2 \right\} f''(x)\,dx$$

where

$x_m = (x_j + x_{j-1})/2$ is the midpoint.

Take the absolute value:

$$|R_1(f\,|\,[x_{j-1}, x_j])| \le K_2(f)\ \frac{1}{2}\int_{x_{j-1}}^{x_j} \left|\left(\frac{b-a}{2n}\right)^2 - (x-x_m)^2\right| dx$$

We can remove the absolute values since $\frac{b-a}{2n} = \frac{1}{2}\Delta x_j$

and $|(x-x_m)| \le \frac{\Delta x_j}{2} \Rightarrow$ the difference is positive.

WA 2-4

$$|R_1(f;[x_{j-1},x_j])| \leq \frac{K_2(f)}{2}\left\{\left(\frac{b-a}{2n}\right)^2 \Delta x_j - \frac{(x-x_M)^3}{3}\Bigg|_{x_{j-1}}^{x_j}\right\}$$

$$\leq \frac{K_2(f)}{2}\left\{\frac{1}{4}\left(\frac{b-a}{n}\right)^3 - \frac{1}{12}\left(\frac{b-a}{n}\right)^3\right\}$$

$$\leq \frac{1}{12}K_2(f)\frac{(b-a)^3}{n^3}$$

Finally, go back to (8a)-(8b) and sum:

$$|R_n(f)| \leq \sum_{j=1}^{n}\left|\frac{\Delta x}{2}\left[f(x_{j-1})+f(x_j)\right] - \int_{x_{j-1}}^{x_j} f(x)\,dx\right|$$

$$\leq \frac{1}{12}K_2(f)\frac{(b-a)^3}{n^2}$$

since the sum gives n. This is the result!