# MA575

(1) Solutions to the Final : F2014

1. $\Leftarrow$ $\exists A$ s.t. $\forall \varepsilon > 0$ $\exists P_\varepsilon$ on $[a,b]$ s.t.

$$|U(f,P_\varepsilon) - A| < \varepsilon \ \& \ |L(f,P_\varepsilon) - A| < \varepsilon.$$

Then replacing $\varepsilon$ by $\varepsilon/2$ we have

$$U(f,P_\varepsilon) - L(f,P_\varepsilon) \leq |U(f,P_\varepsilon) - A| + |L(f,P_\varepsilon) - A| \leq \varepsilon.$$

This is the fund. criteria for integrability

$\Rightarrow$) If $f$ is integrable $\forall \varepsilon > 0$ $\exists P_\varepsilon$ s.t.

$$0 \leq U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon.$$

Set $A = \int_a^b f = \overline{\int_a^b} f = \underline{\int_a^b} f$ since $f$ is integrable.

By defn $A = \inf U(f,P) = \sup L(f,P)$

So given $\varepsilon > 0$ $\exists P_\varepsilon^{(1)}$ s.t. $|U(f,P_\varepsilon^{(1)}) - A| < \varepsilon$

and $\exists P_\varepsilon^{(2)}$ s.t. $|L(f,P_\varepsilon^{(2)}) - A| < \varepsilon$

Let $P_\varepsilon = P_\varepsilon^{(1)} \cup P_\varepsilon^{(2)}$ be a refinement of $P_\varepsilon$.

Then $L(f,P_\varepsilon^{(2)}) \leq L(f,P_\varepsilon) \leq U(f,P_\varepsilon) \leq U(f,P_\varepsilon^{(1)})$

$\Rightarrow$ $|L(f,P_\varepsilon) - A| < \varepsilon$ & $|U(f,P_\varepsilon) - A| < \varepsilon$. ∎

2. $f(x) \leq f(y)$ $\forall x \leq y$ and $f$ is bounded.
For any partition $P$ :

$$U(f,P) = \sum M_i \Delta x_i$$

where $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$

$$L(f,P) = \sum m_i \Delta x_i$$

where $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$

Because $f$ is monotone incr. $M_i = f(x_i)$ & $m_i = f(x_{i-1})$

so

$$U(f,P_\varepsilon) = f(x_1)(x_1 - x_0) + f(x_2)(x_2 - x_1) + \dots + f(x_n)(x_n - x_{n-1})$$

$$L(f,P_\varepsilon) = f(x_0)(x_1 - x_0) + f(x_1)(x_2 - x_1) + \dots + f(x_{n-1})(x_n - x_{n-1})$$

and

$$U(f,P) - L(f,P) = (f(b) - f(a))\Delta x$$

if the interval has uniform length.

Given $\varepsilon > 0$ choose $\Delta x = \dfrac{\varepsilon}{f(b) - f(a)}$ so

with this partition:

$$U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon.$$

Note: We assume $f(b) - f(a) > 0$, i.e. $f$ isn't a constant.

③

3. $f_n \in C(I)$ & $f_n \to f$ unif on $I$

$I \subset \mathbb{R}$ closed & bdd.

$\{x_n\} \subset I$ and $x_n \to x_0 \in I$

Then $\lim_{n\to\infty} f_n(x_n) = f(x_0)$.

*Proof*

$f_n \to f$ uniformly means $\forall \varepsilon > 0 \ \exists N_\varepsilon$ s.t. $n > N_\varepsilon \Rightarrow \|f_n - f\|_{\infty, I} = \sup_{x \in I} |f_n(x) - f(x)| < \varepsilon$

Write: $f_n(x_n) - f(x_n) + f(x_n) - f(x_0) = f_n(x_n) - f(x_0)$

Given $\varepsilon > 0$, $\exists N^{(1)}_\varepsilon$ s.t. $n > N^{(1)}_\varepsilon \Rightarrow \|f_n - f\|_\infty < \varepsilon/2$

$\exists N^{(2)}_\varepsilon$ s.t. $n > N^{(2)}_\varepsilon \Rightarrow |f(x_n) - f(x_0)| < \varepsilon/2$ since $f$ is cont)

let $N_\varepsilon = \max(N^{(1)}_\varepsilon, N^{(2)}_\varepsilon)$

then

$$|f_n(x_n) - f(x_0)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)|$$
$$\leq \|f_n - f\|_{\infty, I} + |f(x_n) - f(x_0)|$$
$$\leq \varepsilon$$

4. Part 1 Apply the Weierstrass M-Test:

let $f_n(x) = \frac{\cos^n x}{n^2}$ on $\mathbb{R}$

$|f_n(x)| \le \frac{1}{n^2}$ on $\mathbb{R}$. Set $M_n = \frac{1}{n^2}$

Note that $\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} \frac{1}{n^2} < \infty$ by the p-test.

Whence by the M-Test the series converges uniformly on $\mathbb{R}$. ∎

Part 2 We know $\sum_{j=0}^{\infty} a_j (x-c)^j$ converges absolutely for $x \in (c-R, c+R)$. Let $I \subset (c-R, c+R)$ be closed $[a,b]$ (for $a, b$ finite so $I$ is compact)

Let $f_j(x) = a_j (x-c)^j$, $x \in [a,b]$

$\exists\, d \in [a,b]$ so $|f_j(x)| \le a_j |d-c|^j$

Since $|d-c| < R$ the series $\sum_{j=0}^{\infty} a_j |d-c|^j$ converges so by the Weierstrass M-Test $\sum_{j=0}^{\infty} a_j (x-c)^j$ converges uniformly on $I$.

Any compact subset of $(c-R, c+R)$ can be put in such an $I$. ∎

(5)

5. $X = C_{\mathbb{R}}([0,1])$ is separable.

Let $f \in C_{\mathbb{R}}([0,1])$

By the Weierstrass Polynomial approx. theorem

given $\varepsilon > 0$ $\exists$ $p(x)$, a polynomial so

$$\|f - p\|_{\infty, [0,1]} < \varepsilon/2$$

Since $p(x) = \sum_{j=1}^{N} a_j x^j$, $a_j \in \mathbb{R}$, $\exists N$ rational

numbers $q_j$ so $|q_j - a_j| < \varepsilon/2N$

Then $\sup_{x \in [0,1]} |p(x) - \sum q_j x^j|$

$= \sup_{x \in [0,1]} |\sum (a_j - q_j) x^j|$

$\leq \varepsilon/2$

Let $\mathcal{D} = \{ p(x) = \sum_{0}^{N} q_j x^j$ w/ $q_j \in \mathbb{Q}, N \in \mathbb{N} \}$

Claim $\mathcal{D}$ is dense in $C_{\mathbb{R}}([0,1])$. Since $\mathcal{D}$

is countable $\Rightarrow C_{\mathbb{R}}([0,1])$ is separable

Pf of Claim: Given $f$ and any $\varepsilon > 0$ choose $p_\varepsilon$ as above w/

$\|f - p_\varepsilon\|_{\infty, [0,1]} < \varepsilon/2$. Choose $\sum q_j x^j$

so $\|f - \sum q_j x^j\| \leq \varepsilon$. ∎