MA681–001 Functional Analysis
Fall 2013
Problem Set 4
DUE: Wednesday, 6 November 2013

(1) Suppose $(A, D(A))$ is densely defined. Prove that $(A^*, D(A^*))$ is closed. Show that $\Gamma(A^*) = V\Gamma(A)^\perp$.
(2) Suppose that $(A, D(A))$ is densely defined and closed. Prove that $(A^{**}, D(A^{**})) = (A, D(A))$. Show that $\Gamma(A^{**}) = V\Gamma(A^*)^\perp$ and relate it to $\Gamma(A)$.
(3) Let $T \in \mathcal{L}(\mathcal{H})$ be a bounded operator on a Hilbert space $\mathcal{H}$. Prove that i) $\|T^*\| = \|T\|$, and ii) $\|T^*T\| = \|T\|^2$.
(4) Prove that an isometry on a finite-dimensional complex Hilber space (isomorphic to $\mathbb{C}^{\mathbb{N}}$, $N < \infty$) is unitary.
(5) Let T_L be the left shift operator on $\ell^2(\mathbb{C})$. Compute the spectrum of T_L. Find the eigenvalues of T_L and the corresponding eigenvectors. What can you say about the spectrum of its adjoint? Does the adjoint have any eigenvalues?
(6) Suppose T_n is a sequence of bounded operators on a Hilbert space $\mathcal{H}$ so that $\langle f, T_n g\rangle$ converges for each pair of vectors $f, g \in \mathcal{H}$. Prove that there exists a bounded operator T so that the sequence T_n converges weakly to T. Hint: use the principle of uniform boundedness twice.