Answers Test 2 **MA 433 S 08**

1. (a) $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$ and $\cos z = \frac{e^{iz}+e^{-iz}}{2}$.
(b) $\sin(\pi/2+i) = \frac{e^{i(\pi/2+i)}-e^{-i(\pi/2+i)}}{2i} =$ (since $e^{\pm\pi i/2} = \pm i$) $= (e^{-1}+e)/2 = \cosh(1)$.
Also, $\cos(\pi-2i) = \frac{e^{i(\pi-2i)}+e^{-i(\pi-2i)}}{2} =$ (since $e^{\pm\pi i} = -1$) $= -(e^2+e^{-2})/2 = -\cosh(2)$.

2. (a) If $z = re^{i\theta}, 0 < r < \infty$ and $-\pi/2 < \theta < 3\pi/2$, let $l(z) = \ln r + i\theta$.
(b) $l(-1) = i\pi$, since $-1 = e^{i\pi}$.
(c) $b(z) = e^{-2il(z)}$ and by the chain rule, $b'(z) = -2ie^{-2il(z)}\, z^{-1} = -(2i/z)b(z)$ when z is in D ($z \in D$). Thus using (b), $b'(-1) = 2ie^{2\pi}$.

3. (a) $\int_\Gamma z\,\cos(z^2)\,dz = (1/2)\,\sin(z^2)|_1^i = -\sin(1)$. by the theorem on evaluating line integrals using antiderivatives.
(b) By Cauchy's Theorem for derivatives: $\int_C \frac{e^{3z}}{(z-\pi i/2)^4}\,dz = \frac{2\pi i}{3!}\,\frac{d^3e^{3z}}{dz^3}\,|_{z=\pi i/2}$
$= (\pi i/3)(27)e^{3\pi i/2} = 9\pi$.
(c) Let $z(t) = e^{it}, 0 \le t \le \pi/2$. Then $\bar z(t) = e^{-it}, dz/dt = ie^{it}$ and
$\int_\gamma \bar z^2\,(z+2)\,dz = i\int_0^{\pi/2} e^{-2it}\,(e^{it}+2)\,e^{it}\,dt = it - 2e^{-it}|_0^{\pi/2} = (\pi/2+2)i+2$

3. Note that $1/P$ is analytic in $\{z : 1 \le |z| \le R\}$ since P has all of its zeros in $B(0,1)$. Thus from the generalized Cauchy theorem (proved in class using the method of crosscuts), we deduce $\int_{\Gamma_1}\frac{dz}{P(z)} = \int_{\Gamma_R}\frac{dz}{P(z)}$.
(b) Observe that $\lim_{|z|\longrightarrow\infty}(|P(z)|/|z|^n) = \lim_{|z|\longrightarrow\infty}|1+a_{n-1}z^{-1}+\ldots+a_0\,z^{-n}| = 1$. Thus there exists R_0 such that $|P(z)| \ge |z|^n/2$ for $|z| \ge R_0$. Another way to get this inequality would be to let $M = \sum_{k=0}^{n-1}|a_k|+1$ and $R_0 = 2M$. If $|z| \ge R_0$, then from the triangle inequality, $|P(z)| \ge |z|^n - |a_{n-1}z^{n-1}+\ldots+a_0| \ge$ (again by the triangle inequality) $\ge |z|^{n-1} - (|a_{n-1}||z|^{n-1}+\ldots+|a_0|) \ge$ (since $|z| \ge R_0 \ge 1$) $\ge |z|^n(1-M/|z|) \ge |z|^n/2$ since $|z| \ge R_0 = 2M$.
(c) From (b), $\left|\int_{\Gamma_R}\frac{dz}{P(z)}\right| \le 2\int_{\Gamma_R}|z|^{-n}\,|dz| = 4\pi R^{1-n}\to 0$ as $R\to\infty$ since $n \ge 2$.
(d) $\int_{\Gamma_2}\frac{dz}{P(z)} = 0$.

4. (a) Fix $z_0 \in \mathbf{C}$ and for $R \ge 1$, let $C_R = z_0 + Re^{it}, 0 \le t \le 2\pi$. If $|f(z)| \le M < \infty$ for all $z \in \mathbf{C}$, then from the Cauchy integral formula for the derivative of f we see that
$f'(z_0) = \frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{(z-z_0)^2}\,dz$. Hence $|f'(z_0)| \le \frac{1}{2\pi}\int_{C_R}\frac{|f(z)|}{|z-z_0|^2}\,|dz| \le \frac{M}{2\pi R^2}\,L(C_R) = M/R$,
where we have used the fact that $|z-z_0| = R$ on C_R. Letting $R\to\infty$, it follows that $f'(z_0) = 0$. Since $z_0 \in \mathbf{C}$ is arbitrary we conclude first that $f' \equiv 0$ in $\mathbf{C}$ and second from classwork that $f \equiv c$ for some complex number c.

Homework Problems to Hand in **MA 433 S 08**

This homework counts as one take home test = 100 points. Show your work as answers are often in the book. You can ask me for hints if you get stuck. You may also consult with others but please do so sparingly. In general do your own work! Please write legibly with a pen or dark pencil. The homework on chapters 5, 6, will be collected one period after we finish each chapter. The homework on chapter 7 will be collected one period after we finish 7.78. The * homework will not be collected.

Section 4.50* (page 171) 6, 7.
Section 5.52 (page 181) 4. **Hint:** Let $z = re^{i\theta}$ and take Real, Imaginary parts of (10), page 180.
Section 5.54 (page 189) 2, 5, 7, 11 (b).
Section 5.56 (page 198) 1, 4, 6.
Section 5.60 (page 212) 1, 6, 7.
Section 5.61 (page 218) 1, 2, 8 (do not find E_6).
Section 6.64 (page 230) 1 (b), (e), 2 (d), 3 (a).
Section 6.65 (page 233) 1 (a), (c), 2 (a).
Section 6.67 (page 238) 1 (b), 2 (b), 6 (c).
Section 7.72 (page 257) 2, 6 (give all limiting arguments).
Section 7.74 (page 265) 1, 4 (give all limiting arguments).
Section 7.77 (page 276) 1, 2, 3 (give all limiting arguments).
Section 7.78 (page 280) 1, 5.
Section 7.80* (page 285) 1 (c), 6 (a).
Section 7.82* (page 296) 2, 4, 6.