Test Review 2 **MA 433** **S08**

The test will be over chapter 3 and chapter 4 up to section 50. The theme of chapter 3 was elementary functions so you should know the definitions, be able to prove analyticity, and compute values of each function studied in this chapter. The theme of chapter 4 is contour integration so you should be adept at finding contour integrals by all the methods we studied in this chapter: (a) directly from the definition, (b) by finding antiderivatives, (c) by using Cauchy's theorem and (d) by using the Cauchy integral formula for an analytic function and its derivatives. Besides the various theorems of Cauchy, be able to state and outline the proof of Morea's theorem, Liouville's theorem, and the fundamental theorem of algebra.

Practice Test 1

15 pts 1. (a) Give the definition of $\sin z, \cos z$ in terms of the complex exponential function. Use your definition to show that $\sin^2 z + \cos^2 z = 1$.

15 pts 2. (a) Give the definition of the inverse sine of z (denoted $\sin^{-1} z$).
(b) What is meant by an analytic branch of $\sin^{-1} z$ in a domain D?
(c) If $f(z)$ is an analytic branch of $\sin^{-1} z$ in a domain D show that $(1 - z^2)\, f'(z)^2 = 1$.
Hint: Use the chain rule and implicit differentiation.

25 pts 3. Find the following as complex numbers:
(a) $e^{(\pi/2)(1+i)}$
(b) $\tan(i)$
(c) $\log(1 + \sqrt{3}i)$
(d) $\frac{d}{dz} PV(z^i)\,|_{z=i}$.

45 pts 4. Find the following integrals:
(a) $\displaystyle\int_C \frac{\text{Log } z}{(z-1)^4}\, dz$ where C is the circle $|z - 1| = 1/2$ oriented counterclockwise.
(b) $\displaystyle\int_\gamma (x - y)\, dz$ where γ has parametrization: $z(t) = e^{it}$ for $\pi/2 \le t \le 3\pi/2$.
(c) $\displaystyle\int_\gamma (x - y)\, |dz|$ where γ has parametrization: $z(t) = e^{it}$ for $\pi/2 \le t \le 3\pi/2$.
(d) $\displaystyle\int_\Gamma e^{z^3}\, dz$ where Γ is the ellipse $20x^2 + 100y^2 = 1$ oriented counterclockwise.

10 pts EC. State the fundamental theorem of algebra and outline its proof. You may assume Liouville's theorem provided you first state this theorem.

Answers Practice Test 1

1. See the book, page 100.

2. (a) w is in $\sin^{-1} z$ if and only if $\sin w = z$. (b) f is an analytic branch of $\sin^{-1} z$ in D provided f is analytic in D and $\sin(f(z)) = z$ whenever z is in D. By the chain rule and implicit differentiation, $1 = \frac{d}{dz}\sin(f(z)) = f'(z)\cos f(z)$. Squaring both sides of this equation and using $\cos^2(f(z)) = 1 - \sin^2(f(z)) = 1 - z^2$, it follows that $(1 - z^2)f'(z)^2 = 1$ for all z in D.

3. (a) $ie^{\pi/2}$ (b) $i\tanh 1 = i\frac{e-e^{-1}}{e+e^{-1}}$ (c) $\{\ln 2 + i(\pi/3 + 2k\pi),\ k = 0, \pm 1, \ldots\}$ (d) $e^{-\pi/2}$.

4. (a) $2\pi i/3$, (b) $\pi(1+i)/2$, (c) - 2, (d) 0.

EC. See the book, section 49.

Practice Test 2

25 pts 1. (a) Find $\tan^{-1} z$ in terms of log ' s.
(b) Use your answer to find $\tan^{-1}(3i)$.
(c) Given that Re $\frac{i+z}{i-z} > 0$ in $B(0,1) = \{z : |z| < 1\}$. Use (a) to define an analytic branch of $\tan^{-1} z$ in $B(0,1)$. Justify your reasoning.

50 pts 2. Find the following integrals:
(a) $\displaystyle\int_G z \sin z \, dz$ where C is the line segment from 0 to 3 followed by the line segment from 3 to -i.
(b) $\displaystyle\int_C \frac{\cos z}{(z^2+1)(z-i)} \, dz$
where C is the circle $\{z : |z| = 2\}$ oriented counterclockwise.

(c) $\displaystyle\int_{C^*} \frac{\cos z}{(z^2+1)(z-i)} \, dz$
where C^* is the circle $\{z : |z - i| = 1\}$ oriented counterclockwise.

25 pts 3. Let D be a simply connected domain and f analytic in D with $f(z) \neq 0$ for z in D.
(a) Does there exist F analytic in D with $F'(z) = f'(z)/f(z)$ when z is in D (justify your reasoning, i.e, quote theorems).
(b) Show that $f\,e^{-F}$ is constant in D so that $f = c\,e^F$ for some complex number $c \neq 0$.
(c) Use (b) to show that f has for a given positive integer n, an analytic n th root k in D. That is, k is analytic in D and $(k(z))^n = f(z)$ for each $z \in D$.

10 pts EC. Find, $\displaystyle\int_\Gamma |z - \tfrac{1}{2}|^{-2} \, |dz|$ where Γ is the circle $\{z : |z| = 1\}$ oriented counterclockwise.
Hint: On Γ, $|dz| = -idz/z$ and $\bar{z} = 1/z$. Now write the modulus in terms of z and $\bar{z}$ and convert the integral to an integral where the Cauchy integral formula can be used.

Answers Practice Test 2

1. (a) $\tan^{-1} z = (i/2)\log\left(\frac{i+z}{i-z}\right)$ (b) $\tan^{-1}(3i) = \{i\ln\sqrt{2} - \pi/2 - k\pi,\ k = 0, \pm1, \ldots\}$ (c) Define $\mathrm{Tan}^{-1}z = (i/2)\ \mathrm{Log}\left(\frac{i+z}{i-z}\right)$. To see that this function is analytic note from the given that $\mathrm{Re}\left(\frac{i+z}{i-z}\right) > 0$, when $|z| < 1$, so never equal to a nonpositive real number. Since $(i/2)$ Log w is analytic in $\mathbb{C}\setminus(-\infty, 0]$ and a composition of analytic functions is analytic, assuming the composition makes sense, it follows that $\mathrm{Tan}^{-1}z$ is analytic in $B(0,1)$.

2. (a) i (cosh 1 - sinh 1).
(b) 8 π / 3.
(c) $-\pi i($ sinh 1).
(d) $\pi i(-$ sinh 1 + (1/2) cosh 1).

3. (a) Note that f'/f is analytic in D since f is nonzero in D. Thus by Cauchy's Theorem $\int_\Gamma f'/f\, dz = 0$ for each closed contour Γ. From classwork or Corollary 1, page 117, vanishing on closed curves is equivalent to the existence of F.
(b) $(f\, e^{-F})' = (f' - fF')e^{-F} = (f' - f')e^{-F} \equiv 0$ in D. Thus from earlier work (see the Theorem on page 56) $fe^{-F} = c$.
(c) Put $k(z) = \omega\, e^{F/n}$ where ω is in $c^{1/n}$ and c is as in (b). Clearly, $k^n = f$.

EC. $8\pi/3$.