Test Review Final **MA 433 S08**

The final exam is comprehensive with more emphasis over the material covered since the last test. As regards the new material, be able to find derivatives and integrals of analytic functions defined by power series within the disk of convergence. Also know how to find the Laurent expansion of an analytic function in an annulus about a given point. Be able to classify the isolated singularities of an analytic function: removable, pole, essential. Know the residue theorem and how to apply it in finding the various integrals in chapter 7. Practice finding such integrals, giving all limiting arguments. As for the old material, be able to work with complex numbers and functions, finding absolute values, conjugates, limits, derivatives, etc. Have at least a general knowledge of the Cauchy Riemann equations and harmonic functions. Review the definitions and analyticity of the exponential, trigometric, hyperbolic, logarithmic, powers, and inverse trigometric functions. Know how to evaluate integrals by all the methods we have studied. Have a working knowledge of the major theorems in this course, including all the Cauchy theorems, Morea's theorem, the fundamental theorem of algebra, Liouville's theorem, the maximum modulus theorem, the residue theorem and the identity theorem for analytic functions. As usual your homework, notes, and old tests should be a good review for the test. Here are two tests similar to the actual final test to practice on.

Practice Final 1

20 pts 1. Find the following
(a) The points $z = x + iy$ where $f'(z)$ exists, when $f(z) = (x-2)^2 + y^2 - y + ix$.
(b) $\lim_{z \longrightarrow 0} z/|z|$ or indicate why the limit does not exist.
(c) The sum of the series, $\displaystyle\sum_{n=1}^{\infty} \frac{(1+i)^n}{(2+i)^n}$.
(d) All solutions to $e^z = \sqrt{3} + i$.
(e) The set of all fourth roots of unity.

15 pts 2. (a) Indicate why Log $(1-z^2)$ is analytic in $B(0,1)$.
(b) Assuming (a), find the Maclaurin series expansion of Log $(1-z^2)$ about $z = 0$.
(c) Find the Maclaurin series expansion of the antiderivative, F, of Log $(1-z^2)$ in $B(0,1)$ with $F(0) = 1$.

10 pts 3. Given $f(z) = e^z \csc z$ and that f has a Laurent series expansion in $0 < |z| < r$.
(a) What is the largest r for which this expansion converges to f in the above annulus? Indicate your reasoning.
(b) If $f(z) = a/z + b + cz + dz^2$, find a, b, c, d.

25 pts 4. Find the value of the following integrals. Indicate your reasoning.
(a) $\displaystyle\int_C \frac{1}{z(z-i)^3}\,dz$ where $C = \{2e^{it} : 0 \le t \le 2\pi\}$.
(b) $\displaystyle\int_\Gamma \frac{1}{z(z-i)^3}\,dz$ where $\Gamma = \{i + (1/2)e^{it} : 0 \le t \le 2\pi\}$.
(c) $\displaystyle\int_0^{2\pi} U(e^{it})dt$ where $U(z) = 2+$ Re $(2e^z + z)$.
Hint: use the mean value theorem for harmonic functions.

20 pts 5. Given f analytic in $\bar{B}(0,R) = \{z : |z| \le R\}$ and that $|f|$ has an absolute maximum in $\bar{B}(0,1)$. (a) What does the maximum modulus theorem say about this absolute maximum?

(b) Given that $\cos w = \sum_{n=0}^{\infty}(-1)^n \frac{w^{2n}}{(2n)!}$. Use (a) and this series, to find the absolute maximum of $f(z) = \cos(2z) - 1$ in $\bar{B}(0, R)$. Indicate your reasoning.

35 pts 6. Use the residue theorem to find the following integrals.

(a) $\int_0^{2\pi} \frac{d\theta}{2 + \sin\theta}$

(b) $\int_{-\infty}^{\infty} \frac{\sin^2(3x)}{1 + x^2}$ **Hint:** $\sin^2(3x) = \text{Re } (1 - e^{i6x})/2$.

10 pts **Extra Credit.** State and prove the identity theorem for analytic functions.

Answers Practice Final 1

1 (a) Only when $z = 2$.
(b) No limit since this function is -1 on the negative real axis and 1 on the positive real axis, so cannot get close to any one number as $z \to 0$.
(c) $1 + i$
(d) $\{\ln 2 + i(\pi/6 + 2k\pi),\ k = 0, \pm 1, \pm 2, \ldots\}$.
(e) $\{\pm 1, \pm i\}$.

2. (a) Analytic because Re $(1 - z^2) > 0$ when $|z| < 1$, Log z is analytic in $\mathbf{C} \setminus (-\infty, 0]$, and because a composition of analytic functions is analytic.

(b) $\text{Log } (1 - z^2) = -\sum_{n=1}^{\infty} \frac{z^{2n}}{n}$ for z in $B(0, 1)$.

(c) $F(z) = 1 - \sum_{n=1}^{\infty} \frac{z^{2n+1}}{n(2n+1)}$ for z in $B(0, 1)$.

3. (a) $r = \pi$ since by a theorem in the book, the Laurent series of f converges to f in any annulus in which f is analytic. Clearly $e^z \csc z$ is analytic in $0 < |z| < \pi$ but not in an annulus of larger radius.

(b) $f(z) = 1/z + 1 + 2z/3 + z^2/3$.

4.(a) 0 by finding the residues at $0, i$ of the integrand
(b) - $2\pi = 2\pi i\cdot$ residue at i of the integrand.
(c) 2π U (0) = 4 π by the mean value theorem for harmonic functions.

5. (a) $2\pi/\sqrt{3}$
(b) $(\pi/2)(1 - e^{-6})$.

6. (a) The absolute maximum of $|f|$ occurs at some point on $\partial B(0, R)$.

(b) $|\cos(2z) - 1| \leq$ (by the triangle inequality) $\sum_{n=1}^{\infty} \frac{|2z|^{2n}}{(2n)!} = \cos(i2|z|) - 1$.

So the maximum of $|\cos(2z) - 1|$ in $\bar{B}(0, R)$ is $\cosh(2R) - 1$.

EC. See class notes.

Practice Final 2

30 pts 1. Find the following
(a) All solutions to the equation $\cos z = 3$.
(b) A harmonic conjugate v of $e^{-2y}\sin(2x)$ in $\mathbb{C}$.
(c) The Taylor series expansion of $(z+1)\,e^z$ about $z_0 = -2$.
(d) The set $(1+i\sqrt{3})^i$ (write out completely).
(e) The image under the mapping $w = e^z$ of $\{z = x+iy : \ln 1 < x < \ln 2, 0 < y < 2\pi\}$.

15 pts 2. (a) What are the rectangular Cauchy Riemann equations for a function $f = u + iv$ at $z_0 = x_0 + iy_0$? What do the Cauchy Riemann equations have to do with analytic functions.
(b) Use the Cauchy Riemann equations to show that if f is analytic in a domain D and $|f|$ is constant in D, then f is constant in D.

20 pts 3. Find the value of the following integrals. Indicate your reasoning.
(a) $\displaystyle\int_C \frac{1}{(z-1)(e^z-1)}\,dz$ where C is the rectangle with vertices at $(\pm 2, \pm 2)$ oriented once counter clockwise.
(b) $\displaystyle\int_C z^{-1}\,dz$ where C is the arc of the ellipse: $z(t) = \cos t + i2\sin t : -\pi/2 \le t \le \pi/2$.

20 pts 4. (a) Find all possible Laurent series expansion about $z_0 = 1$ of $(z^2-1)^{-1}$.
(b) Find the Laurent expansion of $\cos(z+1)/(z+1)^2$ about $z_0 = -1$.

20 pts 5. Find all singularities of the following functions. Classify each singularity as to whether it is (a) removable, (b) a pole or (c) an essential singularity. If the function has a pole at a singularity, state also the order of the pole:
(a) $\csc^2(z)$
(b) $\sin(2/z^2)$
(c) $(\sin(z^2)/z^2)^3$

35 pts 6. Use the residue theorem to find the following integrals. Include all limiting arguments.
(a) $\displaystyle\int_0^\infty \frac{x^{2/3}\,dx}{x^2+1}$
(b) $\displaystyle\int_0^\infty \frac{x^2}{(x^2+9)^2}\,dx$
(c) $\displaystyle\int_0^\infty \frac{\sin 2x}{x}\,dx$

10 pts EC. Let $f(z)$ be analytic in $\{z : |z| < 1\}$ and suppose that $|f(z)| < (1-|z|)^{-1}$ for $|z| < 1$.
(a) Use the Cauchy integral formula for derivatives applied to the contour $\{z : |z| = \rho\}$ oriented counterclockwise to show that if $0 < \rho < 1$, then $|f''(0)| \le 2\,\rho^{-2}\,(1-\rho)^{-1}$.
(b) Find the best estimate for $|f''(0)|$ that (a) gives. That is, find the minimum of the righthand side of the above inequality on (0,1).

Answers Practice Final 2

1. (a) $\{2\pi k \pm i\ln(3+2\sqrt{2}),\ k=0,\pm1,\pm2,\ldots\}$,
(b) $v=-e^{-2y}\cos(2x)+c$
(c) $(z+1)e^z=-e^{-2}+e^{-2}\sum_{n=2}^{\infty}(z+2)^n\,[\frac{1}{(n-1)!}-\frac{1}{n!}]$
(d) $\{\,e^{-\pi/3+2k\pi+i\ln 2},\ k=0\pm1,\pm2,\ldots\}$
(e) $\{z: 1<|z|<2\}\setminus(-2,-1)$.

2. (a) $u_x=v_y$, $u_y=-v_x$ at z_0 are the Cauchy Riemann equations. If $f=u+iv$ is analytic in a domain D, then u,v satisfy the Cauchy Riemann equations. Also if f has continuous first partials and satisfies the Cauchy Riemann equations in a domain D, then f is analytic in D.
(b) If f is analytic in a domain D and $|f|$ is constant in D, then either $f\equiv 0$ in D, so we are done or $f\bar{f}=c$ in D for some nonzero complex constant c. In this case, $f\neq 0$ in D and $\bar{f}=c/f$ is analytic in D. Applying the Cauchy Riemann equations to $f,\bar{f}$ one gets that all partials of u,v with respect to x,y vanish in D. This implies u,v and so f are constant in D by classwork or a theorem in chapter 2.

3. (a) $2\pi i[-1+(e-1)^{-1}]$ by the residue theorem
(b) πi by choosing a branch of the logarithm that is analytic on C and using the theorem on antiderivatives.

4 (a) $\sum_{n=0}^{\infty}\frac{(-1)^n\,(z-1)^{n-1}}{2^{n+1}}$ converges for $|z-1|<2$ and $\sum_{n=0}^{\infty}\frac{(-1)^n\,2^n}{(z-1)^{n+2}}$ converges for $|z-1|>2$.
(b) $\sum_{n=0}^{\infty}(-1)^n\frac{(z+1)^{2n-2}}{(2n)!}$

5. (a) $\csc^2(z)$ has poles of order 2 at each point of $\{\pi n, n=0,\pm1,\pm2,\ldots\}$ and is analytic everywhere else.
(b) $\sin(2/z^2)$ has an esential singularity at 0 and is analytic everywhere else.
(c) $(\sin(z^2)/z^2)^3$ has a removable singularity at 0. If this function is defined to be 1 at 0, then it is an entire function.

6. (a) π
(b) $\pi/12$
(c) $\pi/2$.

EC. (a) Can do in class if asked.
(b) Minimum is $27/2$ at $\rho=2/3$.