

1. Do problems 4-8 in the book on page 345-346.

2. (a) Fix  $p, 1 < p < n$ , and recall that  $B^0(0, 1) = \{x \in \mathbb{R}^n : |x| < 1\}$ . Show that  $h(x) = |x|^{\frac{p-n}{p-1}}$  is a strong solution to the  $p$  Laplace equation in  $B^0(0, 1) - \{0\}$ . That is,

$$Lh = \nabla \cdot (|\nabla h|^{p-2} \nabla h) = |\nabla h|^{p-4} \left[ (p-2) \left( \sum_{i,j=1}^n h_{x_i} h_{x_j} h_{x_i x_j} \right) + |\nabla h|^2 \Delta h \right] = 0.$$

(b) Let  $1 < p < n$  and  $u > 0, u \in C^2(B^0(0, 1))$ , be a strong solution to the  $p$  Laplace equation ( $Lu = 0$ ) in  $B^0(0, 1)$  with  $\nabla u \neq 0$  in  $B^0(0, 1)$ . Show for  $\eta > 0$  that  $w = u^{1+\eta}$  is a strict subsolution to the  $p$  Laplace equation. That is, at points in  $B^0(0, 1)$ ,

$$Lw = \nabla \cdot (|\nabla w|^{p-2} \nabla w) = |\nabla w|^{p-4} \left[ (p-2) \left( \sum_{i,j=1}^n w_{x_i} w_{x_j} w_{x_i x_j} \right) + |\nabla w|^2 \Delta w \right] > 0.$$

(c) Let  $u, h$  be as in (a), (b) and suppose also that  $u \in C(B(0, 1))$ . Show that if  $u < h$  on  $\partial B(0, 1)$ , then  $u < h$  in  $B^0(0, 1)$ .

**Hint:** Use  $w$  for  $\eta > 0$  sufficiently small and note that if  $\nabla w(z) = \nabla h(z)$  at some  $z \in B^0(0, 1)$ , then  $h$  is a solution and  $w$  a subsolution at  $z$  to the same nondivergence form PDE.