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Monotonicity of the cd-index for polytopes

Louis J. Billera[1], **Richard Ehrenborg**[2,⋆]

[1] Department of Mathematics, White Hall, Cornell University, Ithaca, NY 14853-7901, USA (e-mail: billera@math.cornell.edu)
[2] School of Mathematics, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA (e-mail: jrge@math.ias.edu)

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Abstract. We prove that the $\mathbf{cd}$-index of a convex polytope satisfies a strong monotonicity property with respect to the $\mathbf{cd}$-indices of any face and its link. As a consequence, we prove for d-dimensional polytopes a conjecture of Stanley that the $\mathbf{cd}$-index is minimized on the d-dimensional simplex. Moreover, we prove the upper bound theorem for the $\mathbf{cd}$-index, namely that the $\mathbf{cd}$-index of any d-dimensional polytope with n vertices is at most that of $C(n, d)$, the d-dimensional cyclic polytope with n vertices.

1 Introduction

The problem of determining the relations between the numbers of faces of all dimensions in convex polytopes is one that has amused mathematicians for hundreds of years. For the case of 3-dimensional polytopes, this problem was settled more than 90 years ago by Steinitz [31]. Nearly 20 years ago, the problem was completely settled in arbitrary dimension for the mutually dual cases of simplicial and simple polytopes – those with vertices, respectively, facets, in general position [10, 26]. Yet in spite of some progress, and no lack of effort, the general solution remains elusive. See [7] for a brief survey and references to some of the more recent work in this area. In particular, [1] and [6] describe the current incomplete state of knowledge about inequalities for face numbers of 4-dimensional polytopes.

The form of the solution in the simplicial case is of interest here. Rather than working directly with the numbers of faces of each dimension (the

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f-vector), two linearly equivalent derived invariants are considered, the h-polynomial and the g-polynomial. These were introduced in this context by McMullen [22], who showed by means of a shelling argument that the h-polynomial of a simplicial polytope always has nonnegative coefficients and it is maximized, over all simplicial d-dimensional polytopes with n vertices, by the cyclic polytope $C(n, d)$ (the convex hull of n points on the moment curve $(t, t^2, \ldots, t^d)$). The latter is known as the *Upper Bound Theorem* for polytopes, since it implies that for *all* d-dimensional polytopes with n vertices $C(n, d)$ maximizes the number of faces of all dimensions. One key part of the characterization of f-vectors in the simplicial case is the so-called *Generalized Lower Bound Theorem*, which states that over all convex d-dimensional polytopes, the g-polynomial is minimized termwise on the d-dimensional simplex. This fact gives all the linear inequalities that hold for f-vectors of simplicial polytopes.

For the case of general d-dimensional polytopes, there has been some effort to understand more than just the f-vector. The *flag f-vector* is an invariant that includes the full enumerative information about chains of faces in the polytope, and so includes the usual f-vector, reducing to the latter in the case of simple or simplicial polytopes. In [2] all the linear relations holding for flag f-vectors of polytopes (or, more generally, Eulerian partially ordered sets) are obtained via the Euler relations holding for intervals of faces. The $\mathbf{cd}$-index is a derived invariant that efficiently encodes information carried by the flag f-vector [5].

By means of a shelling argument, Stanley [29] showed that the $\mathbf{cd}$-index is termwise nonnegative for a class of objects somewhat more general than convex polytopes. Since the coefficients of the $\mathbf{cd}$-index of the simplex are all positive, this does not establish that the $\mathbf{cd}$-index is minimized over polytopes by the simplices. However, he conjectured more generally that among all Gorenstein* lattices the Boolean algebra has the termwise smallest $\mathbf{cd}$-index [30, Conjecture 2.7]. The zonotopal analogue of Stanley's conjecture was proved by the authors and Readdy [8], namely, among all zonotopes (or more generally all oriented matroids) the cube has the smallest $\mathbf{cd}$-index. Other inequalities giving evidence of this conjecture were given by Ehrenborg and Fox [14].

By extending the methods of Stanley and McMullen, we show here that the $\mathbf{cd}$-index of any convex d-dimensional polytope is termwise as large as that of the d-dimensional simplex, establishing Stanley's conjecture for polytopes. Further, we show the corresponding upper bound theorem for the $\mathbf{cd}$-index: over all d-dimensional polytopes with n vertices, the cyclic polytope $C(n, d)$ has the termwise largest $\mathbf{cd}$-index. Our methods actually produce much stronger lower bounds. We show that the $\mathbf{cd}$-index satisfies a strong version of the submultiplicative property of the g-polynomial proved recently by Braden and MacPherson [12].

In Sect. 2, we give the basic definitions concerning polytopes, Eulerian posets and the $\mathbf{cd}$-index. This includes a brief introduction to the coalgebra notions that come into play in proving the upper bound theorem. Section 3 contains some identities involving the $\mathbf{cd}$-index of the boundary of a regular cellular ball and of the regular cellular sphere obtained by attaching two such balls along their common boundary. We compute the $\mathbf{cd}$-index of partial shellings in Sect. 4 and use this in Sect. 5 to prove the submultiplicative inequalities. Section 6 is dedicated to the proof of the upper bound theorem. Finally, some consequences of these results for inequalities on the flag f-vector are considered in Sect. 7.

2 Polytopes, Eulerian posets and the $\mathbf{cd}$-index

A *partially ordered set* (poset) P is graded if it has a minimal element $\hat{0}$, maximal element $\hat{1}$, and for every element x in the poset, every maximal chain from $\hat{0}$ to x has the same length. Let the rank of an element x, $\rho(x)$, be the length of a maximal chain from $\hat{0}$ to x. We call $\rho(P) = \rho(\hat{0}, \hat{1})$ the rank of the poset P. For $x \leq y$ define $\rho(x, y)$ to be equal to $\rho(y) - \rho(x)$ and define the *interval* from x to y to be set $\{z \ : \ x \leq z \leq y\}$, denoted $[x, y]$. Observe that $[x, y]$ is a graded poset of rank $\rho(x, y)$.

For a graded poset P of rank $d+1$ the flag f-vector is defined as follows. For S a subset of $\{1, 2, \ldots, d\}$ let f_S be the number of chains of P whose ranks are exactly given by the set S. That is,

$$f_S = |\{\hat{0} = x_0 < x_1 < \cdots < x_{k+1} = \hat{1} \ : \ \rho(x_i) = s_i\}|,$$

where $S = \{s_1 < \cdots < s_k\}$. A stepping stone in the study of flag vectors is the flag h-vector. It is given by the invertible relation (and corresponding inverse relation):

$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} f_T \quad \text{and} \quad f_S = \sum_{T \subseteq S} h_T.$$

Hence the flag f-vector and the flag h-vector carry the same information about the poset.

Let $\mathbf{a}$ and $\mathbf{b}$ be two non-commuting variables. For a subset S of $\{1, 2, \ldots, d\}$, define u_S to be the $\mathbf{ab}$-monomial $u_1 \cdots u_d$ where $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$. The $\mathbf{ab}$-*index* of a poset P of rank $d+1$, $\Psi(P)$, is defined by

$$\Psi(P) = \sum_S h_S \cdot u_S,$$

where the sum ranges over all subsets S of $\{1, 2, \ldots, d\}$. Observe the $\mathbf{ab}$-index encodes exactly the same information as the flag h-vector. Moreover $\Psi(P)$ is a homogeneous polynomial of degree d.

Another way to view the $\mathbf{ab}$-index is by assigning a weight to each chain in the poset P. For a chain $c = \{\hat{0} = x_0 < x_1 < \cdots < x_{k+1} = \hat{1}\}$ let the *weight* of the chain c be the product $\mathrm{wt}(c) = w_1 \cdots w_d$, where

$$w_i = \begin{cases} \mathbf{b} & \text{if } i \in \{\rho(x_1), \ldots, \rho(x_k)\}, \\ \mathbf{a} - \mathbf{b} & \text{otherwise.} \end{cases}$$

Then the $\mathbf{ab}$-index is given by the sum

$$\Psi(P) = \sum_c \mathrm{wt}(c),$$

where c ranges over all chains in the poset P.

The *Möbius function* $\mu(x, y)$ is defined for $x, y \in P$ by $\mu(x, x) = 1$ and for $x < y$ in P by $\sum_{x \leq z \leq y} \mu(x, z) = 0$. A poset P is called *Eulerian* if the Möbius function satisfies $\mu(x, y) = (-1)^{\rho(x,y)}$. There are linear relations among the entries of the flag f-vector of an Eulerian poset, called the *generalized Dehn-Sommerville relations*, discovered by Bayer and Billera [2]. Fine observed and Bayer and Klapper [5] proved that when P is Eulerian the $\mathbf{ab}$-index of P can be written in terms of the non-commuting variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}$. The resulting polynomial is called the $\mathbf{cd}$-*index*. In fact, they showed that the $\mathbf{cd}$-index exists for a poset if and only if the flag f-vector of the poset satisfies the generalized Dehn-Sommerville relations. Stanley gave another elementary proof of the existence of the $\mathbf{cd}$-index in [29]; see also the discussion in Sect. 3.

Let $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ be the ring of polynomials in the variables $\mathbf{c}$ and $\mathbf{d}$, and let the degree of $\mathbf{c}$ be 1 and the degree of $\mathbf{d}$ be 2. For a poset P, let P^* denote the *dual* poset. The poset P^* has the same underlying set as P but with the order relation $x \leq_{P^*} y$ if $x \geq_P y$. Similarly, for a $\mathbf{cd}$-monomial $v = v_1 v_2 \cdots v_n$, let $v^* = v_n \cdots v_2 v_1$. By linearity we extend this operation to be an involution on $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$. Observe for an Eulerian poset P we have $\Psi(P^*) = \Psi(P)^*$.

For two $\mathbf{cd}$-polynomials v and w, we define $v \leq w$ if the $\mathbf{cd}$-polynomial $w - v$ has nonnegative coefficients. Observe that comparing $\mathbf{cd}$-polynomials coefficientwise is stronger than comparing $\mathbf{ab}$-polynomials since a polynomial can have nonnegative coefficients as an $\mathbf{ab}$-polynomial but not as a $\mathbf{cd}$-polynomial (for example, $\mathbf{a}^2 + \mathbf{b}^2 = \mathbf{c}^2 - \mathbf{d}$).

An important tool in studying the $\mathbf{cd}$-index is that the $\mathbf{cd}$-index is a coalgebra homomorphism. We give a short explanation here; for basic notions of coalgebras, see [24, 32]. For more information on the coalgebra discussed here, we refer the reader to [17]. We extend the ring $\mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$ to a coalgebra; that is, we enrich the ring with a coproduct Δ, which is a linear map $\Delta : \mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle \longrightarrow \mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle \otimes \mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$. We will use the Sweedler notation for the coproduct; hence for the element $\Delta(w)$ we write $\sum_w w_{(1)} \otimes w_{(2)}$. We define our coproduct Δ by $\Delta(\mathbf{c}) = 2 \cdot 1 \otimes 1$, $\Delta(\mathbf{d}) = \mathbf{c} \otimes 1 + 1 \otimes \mathbf{c}$, and otherwise by the Newtonian condition $\Delta(u \cdot v) = \sum_u u_{(1)} \otimes u_{(2)} \cdot v + \sum_v u \cdot v_{(1)} \otimes v_{(2)}$. Observe that $\mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$ is a coalgebra without a counit.

Let the vector space $\mathcal{E}$ be spanned by all isomorphism types of Eulerian posets of rank greater than or equal to one. Observe that Ψ extends to a linear map from $\mathcal{E}$ to $\mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$. Define a coproduct on $\mathcal{E}$ by $\Delta(P) = \sum_{\hat{0} < x < \hat{1}} [\hat{0}, x] \otimes [x, \hat{1}]$ for an Eulerian poset P and extend by linearity to the space $\mathcal{E}$. Ehrenborg and Readdy [17] proved that the $\mathbf{cd}$-index Ψ is a coalgebra homomorphism from the coalgebra of Eulerian posets $\mathcal{E}$ to the coalgebra $\mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$. We will use this result in the following form. Its importance is that convolution of two linear maps over an interval $[x, z]$ can be computed by only knowing the $\mathbf{cd}$-index $\Psi([x, z])$ and not the whole poset structure of the interval.

Proposition 2.1 *Let L and M be linear maps from $\mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$ into a ring. Then the convolution of L and M on the interval $[x, z]$ of an Eulerian poset is given by*

$$\sum_{x<y<z} L(\Psi([x, y])) \cdot M(\Psi([y, z])) = \sum_w L(w_{(1)}) \cdot M(w_{(2)}),$$

where w is the $\mathbf{cd}$-polynomial $\Psi([x, z])$.

The face lattice of a d-dimensional convex polytope is an Eulerian poset of rank $d + 1$, hence a d-dimensional convex polytope has a $\mathbf{cd}$-index of degree d associated to it. Note that if P is a polytope and $H \subset F$ are faces of P, then the interval $[H, F]$ in the face lattice of P is the face lattice of a convex polytope, denoted F/H. In particular $[\emptyset, F]$ is the face lattice of the face F. Thus, for a polytope P we will write $\Psi(P)$ for $\Psi([\emptyset, P])$ and, more generally, $\Psi(F/H)$ for $\Psi([H, F])$. Also recall that the face lattice of the polar P^* of a polytope P is the dual of the face lattice of P.

The coalgebra techniques in [17] were used to show how the $\mathbf{cd}$-index of convex polytopes changes under certain geometric operations. One of them is essential to us. On the ring $\mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$ define a derivation G by letting $G(\mathbf{c}) = \mathbf{d}$ and $G(\mathbf{d}) = \mathbf{cd}$. Also define a linear operator Pyr on the ring $\mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$ by

$$\text{Pyr}(w) = w \cdot \mathbf{c} + G(w).$$

It is straightforward to check that

$$\text{Pyr}(u \cdot v) = G(u) \cdot v + u \cdot \text{Pyr}(v). \tag{2.1}$$

It is proved in [17] that:

Theorem 2.2 (Ehrenborg-Readdy) *For a polytope P let* $\mathrm{Pyr}(P)$ *denote the* pyramid *over P, that is, the convex hull of a point v not in the affine span of P with the polytope P. Then the* $\mathbf{cd}$*-index of* $\mathrm{Pyr}(P)$ *is given by*

$$\Psi(\mathrm{Pyr}(P)) = \mathrm{Pyr}(\Psi(P)).$$

Since the pyramid operation commutes with polarity, we have that for a $\mathbf{cd}$-polynomial w that $\mathrm{Pyr}(w)^* = \mathrm{Pyr}(w^*)$.

A polytope is said to be *simplicial* if every facet is a simplex. The flag f-vector for simplicial polytopes depends only on the f-vector or, equivalently, the h-vector (see, for example, [3, §7]). Stanley [29] gave the $\mathbf{cd}$-index of simplicial polytopes in terms of the h-vector:

$$\Psi(P) = \sum_{i=0}^{d} h_i \cdot \check{\Phi}_{d,i},$$

where the $\check{\Phi}_{d,i}$ are $\mathbf{cd}$-polynomials. The polynomials $\check{\Phi}_{d,i}$ satisfy the following recursion

$$\check{\Phi}_{d+1,i+1} = G(\check{\Phi}_{d,i}), \tag{2.2}$$

with the boundary conditions:

$$\check{\Phi}_{0,0} = 1 \quad \text{and} \quad \check{\Phi}_{d+1,0} = \sum_{i=0}^{d} \check{\Phi}_{d,i} \cdot \mathbf{c}.$$

Hence we have that $\check{\Phi}_{d,i}$ are nonnegative $\mathbf{cd}$-polynomials. The recursion (2.2) is due to Ehrenborg and Readdy [17]. There are many other recursions for $\check{\Phi}_{d,i}$; see [15], or [19], where there is also a combinatorial interpretation for these polynomials.

3 The boundary of a cellular ball

Let P be a graded poset and let $x < z$ be two elements in P. Using the chain definition for the $\mathbf{ab}$-index and conditioning on the largest element in a chain, one obtains that the $\mathbf{ab}$-index of the interval $[x, z]$ is given by

$$\Psi([x,z]) = (\mathbf{a}-\mathbf{b})^{\rho(x,z)-1} + \sum_{x<y<z} \Psi([x,y]) \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{\rho(y,z)-1}. \tag{3.1}$$

By multiplying on the right with $\mathbf{a}-\mathbf{b}$ and bringing the term $\Psi([x,z]) \cdot \mathbf{b}$ to the right-hand side one obtains

$$\Psi([x,z]) \cdot \mathbf{a} = (\mathbf{a}-\mathbf{b})^{\rho(x,z)} + \sum_{x<y\leq z} \Psi([x,y]) \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{\rho(y,z)}. \tag{3.2}$$

Define three functions f, g and h in the incidence algebra of P by

$$f(x,y) = \begin{cases} \Psi([x,y]) \cdot \mathbf{a} & \text{if } x < y, \\ 1 & \text{if } x = y, \end{cases} \qquad g(x,y) = \begin{cases} \Psi([x,y]) \cdot \mathbf{b} & \text{if } x < y, \\ 1 & \text{if } x = y, \end{cases}$$

and $h(x,y) = (\mathbf{a} - \mathbf{b})^{\rho(x,y)}$. Then (3.2) can be written as $f = g \cdot h$ where the product is the convolution of the incidence algebra. Observe that h is invertible and its inverse h^{-1} is given by $h^{-1}(x,y) = \mu(x,y) \cdot (\mathbf{a} - \mathbf{b})^{\rho(x,y)}$, where $\mu(x,y)$ denotes the Möbius function of the interval $[x,y]$. By expanding the equivalent relation $g = f \cdot h^{-1}$ we obtain

$$\Psi([x,z]) \cdot \mathbf{b} = \mu(x,z) \cdot (\mathbf{a} - \mathbf{b})^{\rho(x,z)} + \sum_{x<y\leq z} \Psi([x,y]) \cdot \mathbf{a} \cdot \mu(y,z) \cdot (\mathbf{a} - \mathbf{b})^{\rho(y,z)}.$$

By moving the term $\Psi[(x,z)] \cdot \mathbf{a}$ to the left-hand side of the equation and cancelling a factor of $\mathbf{b} - \mathbf{a}$ on the right we have:

$$\Psi([x,z]) = -\mu(x,z) \cdot (\mathbf{a} - \mathbf{b})^{\rho(x,z)-1} - \sum_{x<y<z} \Psi([x,y]) \cdot \mathbf{a} \cdot \mu(y,z) \cdot (\mathbf{a} - \mathbf{b})^{\rho(y,z)-1}. \quad (3.3)$$

Equation (3.3) is an alternative recursion for the $\mathbf{ab}$-index, which may be viewed as dual to (3.1). We remark that the trick of dividing with the factor of $\mathbf{b} - \mathbf{a}$ is essentially due to Gábor Hetyei (unpublished). We obtain the existence of the $\mathbf{cd}$-index for an Eulerian poset by adding equations (3.1) and (3.3), using that $\mu(y,z) = (-1)^{\rho(y,z)}$, and recognizing the terms as $\mathbf{cd}$-polynomials. This discussion is the essential step in Stanley's proof of the existence of the $\mathbf{cd}$-index; see the proof of Theorem 1.1 in [29].

We define $\mathbf{cd}$-polynomials α_n and β_n for $n \geq 0$ by $\alpha_0 = -1$ and otherwise by

$$\alpha_{2k} = -\frac{1}{2}\left[(\mathbf{c}^2 - 2\mathbf{d})^k + \mathbf{c} \cdot (\mathbf{c}^2 - 2\mathbf{d})^{k-1} \cdot \mathbf{c}\right],$$
$$\alpha_{2k+1} = \frac{1}{2}\left[(\mathbf{c}^2 - 2\mathbf{d})^k \cdot \mathbf{c} + \mathbf{c} \cdot (\mathbf{c}^2 - 2\mathbf{d})^k\right],$$

$$\beta_{2k} = (\mathbf{c}^2 - 2\mathbf{d})^k \quad \text{and} \quad \beta_{2k+1} = -\mathbf{c} \cdot (\mathbf{c}^2 - 2\mathbf{d})^k.$$

The reverses of the $\mathbf{cd}$-polynomials β_n, namely β_n^*, were used in [16] to see how the $\mathbf{cd}$-index of a polytope changes when cutting off a face, while their negatives were used in the existence proof in [29]. As $\mathbf{ab}$-polynomials we have the identity for β_n:

$$\beta_n \cdot (\mathbf{a} - \mathbf{b}) = ((-1)^n \cdot \mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})^n.$$

Together the α_n and β_n satisfy the recurrences

$$\alpha_n = \beta_{n-2} \cdot \mathbf{d} - \alpha_{n-1} \cdot \mathbf{c} \qquad \text{for } n \geq 2, \tag{3.4}$$
$$\beta_n = \beta_{n-1} \cdot \mathbf{c} - 2 \cdot \alpha_n \qquad \text{for } n \geq 1. \tag{3.5}$$

The first recurrence shows that the α_n also have integer coefficients. Let us also mention that the α_n also satisfy the following recurrence

$$\alpha_{n+1} = G(\alpha_n) - \mathbf{c} \cdot \alpha_n,$$

even though we will not use it.

Let Γ be a finite regular cell (or CW) complex, for instance a polyhedral complex, such that its underlying space $|\Gamma|$ is a topological ball of arbitrary dimension. (We shall call such a complex a *regular cellular ball* from now on.) Let $P = P(\Gamma)$ be the poset of nonempty cells of Γ, where $\tau \leq \sigma$ if $\overline{\tau} \subseteq \overline{\sigma}$, and $\widehat{P}$ be the poset P with minimum and maximum elements $\hat{0}$ and $\hat{1}$ adjoined. It follows directly from Proposition 3.8.9 of [27] that

$$\mu_{\hat{P}}(y, \hat{1}) = \begin{cases} (-1)^{\rho(y,\hat{1})} & \text{if } y \in \operatorname{int}(\Gamma), \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

We now can give an expression for the $\mathbf{ab}$-index of the boundary of Γ in terms of the $\mathbf{cd}$-indices of cells in the interior.

Proposition 3.1 *Let Γ be a regular cellular ball. Then*

$$\Psi(\partial\Gamma) = \sum_{F \in \operatorname{int}(\Gamma)} \Psi(F) \cdot \beta_{\rho(F,\hat{1})-1}.$$

Proof. Let $P = P(\Gamma)$ and $\widehat{P}$ be as above. Equating the expressions for $\Psi(\widehat{P}) = \Psi([\hat{0}, \hat{1}])$ given by (3.1) and (3.3), and using (3.6), we get

$$\begin{aligned}(\mathbf{a}-\mathbf{b})^{\rho(\widehat{P})-1} &+ \sum_{\hat{0}<F<\hat{1}} \Psi(F) \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{\rho(F,\hat{1})-1} \\ &= - \sum_{F \in \operatorname{int}(\Gamma)} \Psi(F) \cdot (-1)^{\rho(F,\hat{1})} \cdot \mathbf{a} \cdot (\mathbf{a}-\mathbf{b})^{\rho(F,\hat{1})-1}.\end{aligned}$$

Moving all the interior terms to the right-hand side we obtain

$$\begin{aligned}&(\mathbf{a}-\mathbf{b})^{\rho(\widehat{P})-1} + \sum_{F \in \partial\Gamma} \Psi(F) \cdot \mathbf{b} \cdot (\mathbf{a}-\mathbf{b})^{\rho(F,\hat{1})-1} \\ &= \sum_{F \in \operatorname{int}(\Gamma)} \Psi(F) \cdot \Big((-1)^{\rho(F,\hat{1})-1} \cdot \mathbf{a} - \mathbf{b}\Big) \cdot (\mathbf{a}-\mathbf{b})^{\rho(F,\hat{1})-1} \\ &= \sum_{F \in \operatorname{int}(\Gamma)} \Psi(F) \cdot \beta_{\rho(F,\hat{1})-1} \cdot (\mathbf{a}-\mathbf{b}).\end{aligned}$$

Considering the face poset of $\partial\Gamma$ and the expression of its $\mathbf{cd}$-index by (3.1), we see that the left-hand side of the last equation is equal to $\Psi(\partial\Gamma)\cdot(\mathbf{a}-\mathbf{b})$, since the face poset of $\partial\Gamma$ has rank one less than that of Γ. By cancelling a factor of $\mathbf{a}-\mathbf{b}$ on both sides of the last equation, the result follows.

Corollary 3.2 *Let P be a polytope and F a nontrivial face of P. Let $F_1,\ldots,F_r$ be all the facets of the polytope P that contains the face F. Then*

$$\Psi(\partial(F_1\cup\cdots\cup F_r)) = \sum_{F\le x<P} \Psi(x)\cdot\beta_{\rho(x,P)-1}.$$

Proof. Observe that, by Lemma 2 and Proposition 2 of [13], $F_1\cup\cdots\cup F_r$ is a regular cellular ball whose interior faces are exactly those faces of the polytope P which contain the face F.

If Γ is a regular cellular d-dimensional ball, then denote by Γ' the regular cell complex obtained from Γ by attaching a single new d-dimensional cell τ along $\partial\Gamma$, that is, such that $\partial\Gamma=\partial\tau$. Note that $|\Gamma'|$ is a d-dimensional sphere. The proof of Lemma 6.3 in [17] also proves the following.

Lemma 3.3 *Let Γ and Λ be regular cellular d-dimensional balls such that $\partial\Gamma=\partial\Lambda=\Gamma\cap\Lambda$. Then*

$$\Psi(\Gamma\cup\Lambda)=\Psi(\Gamma')+\Psi(\Lambda')-\Psi(\Gamma\cap\Lambda)\cdot\mathbf{c}.$$

4 Shelling

We discuss in this section the effect of shelling on the $\mathbf{cd}$-index of a polytope. Stanley [29] makes use of a property of Eulerian regular cellular complexes (for example, of regular cellular subdivisions of a sphere) called S-shellability. He observed that for polytopes that the “line shellings” of Bruggesser and Mani [13] are always S-shellings as well as shellings in the classical sense (C-shellings). (For simplicial complexes as well as cubical complexes, S-shellability and C-shellability are equivalent; for a proof of the latter, see [15].)

Although we will prove some of our results in the generality of regular cellular complexes, our crucial shelling arguments will be restricted to polytopes, and in this case shellings will always refer to line shellings. In particular, we need that for every pair of faces $G\subset F$, there exist a (line) shelling of F such that all the facets of F containing G appear as an initial segment.

We state first a corollary to the proof of [29, Theorem 2.2] that will be useful in what follows. To keep the exposition simpler, we refer the interested reader to [29] for the definition of S-shelling.

Proposition 4.1 (Stanley) *Let $F_1, \ldots, F_n$ be an S-shelling of a regular cellular sphere Ω. Then*

$$0 \leq \Psi(F_1') \leq \Psi((F_1 \cup F_2)') \leq \cdots \leq \Psi((F_1 \cup \cdots \cup F_{n-1})') = \Psi(\Omega).$$

Moreover $\Psi(F_1') = \Psi(F_1) \cdot \mathbf{c}$.

For a d-dimensional regular cellular sphere Ω, the coefficient of the term $\mathbf{c}^{d-2}\mathbf{d}$ is the number of facets minus two. Hence we observe that each inequality in Proposition 4.1 is not an equality.

Lemma 4.2 *Let $F_1, \ldots, F_n$ be a line shelling of a polytope P. For $2 \leq r \leq n-1$ and $\Lambda = (F_1 \cup \cdots \cup F_{r-1}) \cap F_r$ we have*

$$\Psi((F_1 \cup \cdots \cup F_r)') - \Psi((F_1 \cup \cdots \cup F_{r-1})') = \Psi(F_r) \cdot \mathbf{c} - \Psi(\Lambda') \cdot \mathbf{c} + \Psi(\partial \Lambda) \cdot \mathbf{d}.$$

Proof. Since line shellings are reversible, both Λ and $\Gamma = (F_{r+1} \cup \cdots \cup F_n) \cap F_r$ are regular cellular d-dimensional balls. Further $\partial F_r = \Lambda \cup \Gamma$ and $\partial \Gamma = \partial \Lambda = \Gamma \cap \Lambda$. Thus by Lemma 3.3,

$$\Psi(F_r) - \Psi(\Lambda') = \Psi(\Gamma') - \Psi(\partial \Gamma) \cdot \mathbf{c}. \tag{4.1}$$

Let $\breve{\Psi} := \Psi((F_1 \cup \cdots \cup F_r)') - \Psi((F_1 \cup \cdots \cup F_{r-1})')$. By [29, Lemma 2.1],

$$\breve{\Psi} = \big(\Psi(\Gamma') - \Psi(\partial \Gamma) \cdot \mathbf{c}\big) \cdot \mathbf{c} + \Psi(\partial \Gamma) \cdot \mathbf{d}. \tag{4.2}$$

Substitution of (4.1) into (4.2) completes the proof.

Proposition 4.3 *Let $F_1, \ldots, F_n$ be a line shelling of a polytope P. Then for $1 \leq r \leq n-1$ we have*

$$\Psi((F_1 \cup \cdots \cup F_r)') = \sum_F \Psi(F) \cdot \alpha_{\rho(F,P)}$$

where the sum is over all nonempty intersections of $F_1, \ldots, F_r$.

Proof. We proceed by induction on r. The case $r = 1$, that $\Psi(F_1') = \Psi(F_1) \cdot \mathbf{c}$, is included in Proposition 4.1. For the inductive step, we have by Lemma 4.2, Proposition 3.1 and the inductive hypothesis that

$$\begin{aligned}
\Psi((F_1 \cup \cdots \cup F_r)') &= \Psi((F_1 \cup \cdots \cup F_{r-1})') + \Psi(F_r) \cdot \mathbf{c} \\
&\quad - \Psi(\Lambda') \cdot \mathbf{c} + \Psi(\partial \Lambda) \cdot \mathbf{d} \\
&= \sum_F \Psi(F) \cdot \alpha_{\rho(F,P)} + \Psi(F_r) \cdot \alpha_1 \\
&\quad + \sum_F \Psi(F) \cdot \left(-\alpha_{\rho(F,P)-1} \cdot \mathbf{c} + \beta_{\rho(F,P)-2} \cdot \mathbf{d}\right),
\end{aligned}$$

where the first sum on the right is over all nonempty intersections of $F_1, \ldots, F_{r-1}$, and the second is over all nonempty intersections of $F_1, \ldots, F_r$ lying in F_r. The proof is completed by use of the recurrence (3.4).

Similar to Corollary 3.2 we have

Corollary 4.4 *Let P be a polytope and F a nontrivial face of P. Let $F_1, \ldots, F_r$ be all the facets of the polytope P that contain the face F. Then*

$$\Psi((F_1 \cup \cdots \cup F_r)') = \sum_{F \leq x < P} \Psi(x) \cdot \alpha_{\rho(x,P)}.$$

Proof. We may reorder the facets $F_1, \ldots, F_r$ such that they form the initial segment of a line shelling of the polytope P. Then the faces x containing F are precisely intersections of the facets containing F. Now the equality follows from Proposition 4.3.

Corollary 4.5 *Let P be a polytope and F a nonempty face of P. Then*

$$\Psi(P) \geq \sum_{F \leq x < P} \Psi(x) \cdot \alpha_{\rho(x,P)}.$$

Proof. The case when the face F is the polytope P follows from the nonnegativity of the **cd**-index for polytopes. Hence consider the case when the face F is a nontrivial face. Let $F_1, \ldots, F_r$ be all the facets of the polytope P that contain the face F. Then by Proposition 4.1 we have the inequality

$$\Psi(P) \geq \Psi((F_1 \cup \cdots \cup F_r)').$$

Now the result follows from Corollary 4.4.

Proposition 4.6 *Let $[x, z]$ be an interval in an Eulerian poset. Then*

$$\begin{aligned} \mathrm{Pyr}(\Psi([x,z])) - \alpha_{\rho(x,z)} &= \sum_{x<y<z} \mathrm{Pyr}(\Psi([x,y])) \cdot \alpha_{\rho(y,z)} \\ &= \sum_{x<y<z} \alpha_{\rho(x,y)} \cdot \mathrm{Pyr}(\Psi([y,z])). \end{aligned}$$

Proof. We prove the first identity in the case in which the interval $[x, z]$ is the face lattice of a polytope P. Let Q be the pyramid over P with apex

v, and let $F_1, \ldots, F_r$ be a partial shelling of Q consisting of all the faces containing v. Then by Corollary 4.4,

$$\begin{aligned}\Psi(Q) &= \Psi((F_1 \cup \cdots \cup F_r)') \\ &= \sum_{v \leq G < Q} \Psi(G) \cdot \alpha_{\rho(G,Q)} \\ &= \alpha_{\rho(P)} + \sum_{\emptyset < F < P} \mathrm{Pyr}(\Psi(F)) \cdot \alpha_{\rho(F,P)},\end{aligned}$$

since every face G of Q containing v is a pyramid over a face of P.

The general case now follows since the flag f-vectors of polytopes linearly span the flag f-vectors of all Eulerian posets [2,9]. The second identity follows by duality and $\alpha_n^* = \alpha_n$.

5 The main theorem

We can now prove the main result of this paper which leads to a proof of the conjecture of Stanley for polytopes.

Theorem 5.1 *For polytope P and nontrivial face F of P,*

$$\Psi(P) \geq \Psi(F) \cdot \mathrm{Pyr}(\Psi(P/F)), \tag{5.1}$$

$$\Psi(P) \geq \mathrm{Pyr}(\Psi(F)) \cdot \Psi(P/F). \tag{5.2}$$

Proof. It is only necessary to prove the first inequality; the second will follow by duality. Let x be a face of P such that $F < x < P$. By applying Corollary 4.5 to the polytope x, we have that

$$\Psi(x) \geq \sum_{F \leq y < x} \Psi(y) \cdot \alpha_{\rho(y,x)}.$$

Since $\Psi([x, P]) \geq 0$ we have that $\mathrm{Pyr}(\Psi([x, P])) \geq 0$. Thus

$$\Psi(x) \cdot \mathrm{Pyr}(\Psi([x, P])) \geq \sum_{F \leq y < x} \Psi(y) \cdot \alpha_{\rho(y,x)} \cdot \mathrm{Pyr}(\Psi([x, P])).$$

Now sum over all $F < x < P$.

$$\begin{aligned}&\sum_{F < x < P} \Psi(x) \cdot \mathrm{Pyr}(\Psi([x, P])) \\ &\geq \sum_{F < x < P} \sum_{F \leq y < x} \Psi(y) \cdot \alpha_{\rho(y,x)} \cdot \mathrm{Pyr}(\Psi([x, P])) \\ &= \sum_{F \leq y < P} \Psi(y) \cdot \sum_{y < x < P} \alpha_{\rho(y,x)} \cdot \mathrm{Pyr}(\Psi([x, P])) \\ &= \sum_{F \leq y < P} \Psi(y) \cdot \left(\mathrm{Pyr}(\Psi([y, P])) - \alpha_{\rho(y,P)}\right),\end{aligned}$$

where the last step is by Proposition 4.6. Now by cancelling terms we obtain

$$\sum_{F \le x < P} \Psi(x) \cdot \alpha_{\rho(x,P)} \ge \Psi(F) \cdot \mathrm{Pyr}(\Psi([F,P])).$$

Apply now Corollary 4.5 and we have the desired result.

The following is Conjecture 11.1 of [17].

Corollary 5.2 *For any polytope P and facet F of P,*

$$\Psi(P) \ge \mathrm{Pyr}(\Psi(F)),$$

and so among polytopes having F as a facet, $\mathrm{Pyr}(F)$ *has the smallest* $\mathbf{cd}$-*index.*

Proof. This follows from the second inequality in Theorem 5.1 since $\Psi(P/F) = 1$.

Using repeated applications of Corollary 5.2 and the fact the $\mathrm{Pyr}(\cdot)$ is a nonnegative operator on $\mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$, we have proved the conjecture of Stanley [30, Conjecture 2.7] in the case of polytopes.

Theorem 5.3 *For a d-dimensional polytope P, $\Psi(P) \ge \Psi(\Delta_d)$, that is, the* $\mathbf{cd}$-*index for d-dimensional polytopes is minimized on the d-dimensional simplex.*

We also remark that equality in Theorem 5.1 only happens when the polytope P is a pyramid over another polytope Q. In equation (5.1) this occurs when the face F is the apex vertex, while in equation (5.2) when F is the base polytope Q.

Since $w \cdot \mathbf{c} \le \mathrm{Pyr}(w)$ for a nonnegative $\mathbf{cd}$-polynomial w, we conclude the following.

Corollary 5.4 *For any polytope P and nontrivial face F of P, the* $\mathbf{cd}$-*index $\Psi(P)$ is larger than each of the* $\mathbf{cd}$-*polynomials*

$$\mathbf{c} \cdot \Psi(F) \cdot \Psi(P/F), \quad \Psi(F) \cdot \mathbf{c} \cdot \Psi(P/F) \quad \text{and} \quad \Psi(F) \cdot \Psi(P/F) \cdot \mathbf{c}.$$

The following corollary was pointed out to us by Margaret Readdy.

Corollary 5.5 (Readdy-Stanley) *Let P be a d-dimensional polytope. The flag h-vector of the polytope P obtains its maximum value only at the two sets*

$$S = \{1,3,5,\ldots\} \cap \{1,2,\ldots,d\} \quad \textit{and} \quad S = \{2,4,6,\ldots\} \cap \{1,2,\ldots,d\}.$$

This follows since we know that each coefficient of the $\mathbf{cd}$-index of the polytope is positive. For more details, see [25].

The results of this section were motivated in part by the *submultiplicative* inequality for the g-polynomial,

$$g(P) \geq g(F) \cdot g(P/F), \tag{5.3}$$

originally conjectured by Kalai [20] and proved (for rational polytopes) by Braden and MacPherson [12]. This inequality applies to a generalization of the g-polynomial to all d-dimensional polytopes that comes from the study of toric varieties and their intersection homology [28] and that reduces to the simplicial g-polynomial in the simplicial case. The g-polynomial has coefficients expressible in terms of the flag f-vector [5, Theorem 6] (see also [4] and [11, §4] for various ways of computing this dependence). While it carries only a fraction of the information in the $\mathbf{cd}$-index, (5.3) seems to say something much deeper about the structure of polytopes than Theorem 5.1. We note that (5.3) remains true when altered by taking a pyramid over either F or P/F (since this does not change the g-polynomial).

6 The upper bound theorem

Here we prove the $\mathbf{cd}$-version of the upper bound theorem for polytopes. It will allow us to reach similar conclusions about the flag h-vector (and the flag f-vector). We begin by proving certain identities on Eulerian posets. In order to do this, we introduce two linear maps on the algebra of $\mathbf{cd}$-polynomials.

Define $\alpha, \beta : \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle \longrightarrow \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$ by defining them on a monomial w by

$$\alpha(w) = \begin{cases} \alpha_{n+1} & \text{if } w = \mathbf{c}^n, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \beta(w) = \begin{cases} \beta_n & \text{if } w = \mathbf{c}^n, \\ 0 & \text{otherwise,} \end{cases}$$

and extending the definitions by linearity. Observe that the linear map α increases the degree by 1, whereas β does not change the degree. Also note for an interval $[x, z]$ we have $\alpha(\Psi([x, z])) = \alpha_{\rho(x,z)}$ and $\beta(\Psi([x, z])) = \beta_{\rho(x,z)-1}$. The two recursions (3.4) and (3.5) can now be written as

$$\alpha(w \cdot \mathbf{c}) = \beta(w) \cdot \mathbf{d} - \alpha(w) \cdot \mathbf{c}, \tag{6.1}$$
$$\beta(w \cdot \mathbf{c}) = \beta(w) \cdot \mathbf{c} - 2 \cdot \alpha(w), \tag{6.2}$$

for any $\mathbf{cd}$-polynomial w.

Lemma 6.1 *For any $\mathbf{cd}$-polynomial w we have*

$$\sum_w \alpha(w_{(1)}) \cdot w_{(2)} = w - \beta(w).$$

Proof. By linearity it is enough to prove this identity for any $\mathbf{cd}$-monomial w. We do this by induction on w. The base case $w = 1$ is straightforward. Consider now the case when $w = v \cdot \mathbf{c}$. Then $\Delta(w) = 2 \cdot v \otimes 1 + \sum_v v_{(1)} \otimes v_{(2)} \cdot \mathbf{c}$. Hence we have

$$\begin{aligned} 2 \cdot \alpha(v) + \sum_v \alpha(v_{(1)}) \cdot v_{(2)} \cdot \mathbf{c} &= 2 \cdot \alpha(v) + (v - \beta(v)) \cdot \mathbf{c} \\ &= v \cdot \mathbf{c} - \beta(v \cdot \mathbf{c}), \end{aligned}$$

where the last step is by (6.2). For the other case $w = v \cdot \mathbf{d}$ observe that $\Delta(w) = v \cdot \mathbf{c} \otimes 1 + v \otimes \mathbf{c} + \sum_v v_{(1)} \otimes v_{(2)} \cdot \mathbf{d}$. Thus we have

$$\begin{aligned} & \alpha(v \cdot \mathbf{c}) + \alpha(v) \cdot \mathbf{c} + \sum_v \alpha(v_{(1)}) \cdot v_{(2)} \cdot \mathbf{d} \\ &= \alpha(v \cdot \mathbf{c}) + \alpha(v) \cdot \mathbf{c} + (v - \beta(v)) \cdot \mathbf{d} \\ &= v \cdot \mathbf{d} \\ &= v \cdot \mathbf{d} - \beta(v \cdot \mathbf{d}), \end{aligned}$$

where the second step is by (6.1). Hence the induction is complete.

Now we can prove two essential identities.

Proposition 6.2 *Let $[x, z]$ be an interval in an Eulerian poset. Then*

$$\sum_{x<y<z} \alpha_{\rho(x,y)} \cdot \Psi([y, z]) = \Psi([x, z]) - \beta_{\rho(x,z)-1},$$

$$\sum_{x<y<z} G(\alpha_{\rho(x,y)}) \cdot \Psi([y, z]) = \alpha_{\rho(x,z)} - \mathrm{Pyr}(\beta_{\rho(x,z)-1}).$$

Proof. To prove the first identity apply Lemma 6.1 to the $\mathbf{cd}$-polynomial $\Psi([x, z])$. Then the right-hand side of the lemma is $\Psi([x, z]) - \beta_{\rho(x,z)-1}$. The left-hand side evaluates to $\sum_{x<y<z} \alpha_{\rho(x,y)} \cdot \Psi([y, z])$ by Proposition 2.1 using the linear map L to be α and M to be the identity map.

To prove the second identity apply the operator Pyr to the first identity and use the identity (2.1). We then obtain

$$\begin{aligned} & \sum_{x<y<z} \big(G(\alpha_{\rho(x,y)}) \cdot \Psi([y, z]) + \alpha_{\rho(x,y)} \cdot \mathrm{Pyr}(\Psi([y, z]))\big) \\ &= \mathrm{Pyr}(\Psi([x, z])) - \mathrm{Pyr}(\beta_{\rho(x,z)-1}). \end{aligned}$$

Now subtract the second equation of Proposition 4.6 from this identity to get

$$\begin{aligned} \sum_{x<y<z} G(\alpha_{\rho(x,y)}) \cdot \Psi([y, z]) &= \mathrm{Pyr}(\Psi([x, z])) - \mathrm{Pyr}(\beta_{\rho(x,z)-1}) \\ & \quad - \mathrm{Pyr}(\Psi([x, z])) + \alpha_{\rho(x,z)} = \alpha_{\rho(x,z)} - \mathrm{Pyr}(\beta_{\rho(x,z)-1}), \end{aligned}$$

which is the desired equality.

Proposition 6.3 *Let P be a polytope and F a nontrivial face of P. Let $F_1, \ldots, F_r$ be the facets of P that contain the face F. Then we have that*

$$\mathrm{Pyr}(\Psi(\partial(F_1 \cup \cdots \cup F_r))) \geq \Psi((F_1 \cup \cdots \cup F_r)').$$

Proof. Let x be a face of P such that $F \leq x < P$. By Corollary 4.5 we have that

$$\Psi(x) \geq \sum_{F \leq y < x} \Psi(y) \cdot \alpha_{\rho(y,x)}.$$

Since the linear operator G preserves weak inequalities and that $\Psi([x,P]) \geq 0$, we have

$$G(\Psi(x)) \cdot \Psi([x,P]) \geq \sum_{F \leq y < x} G(\Psi(y) \cdot \alpha_{\rho(y,x)}) \cdot \Psi([x,P]).$$

Now sum over all $F \leq x < P$.

$$\begin{aligned}
&\sum_{F \leq x < P} G(\Psi(x)) \cdot \Psi([x,P]) \\
&\geq \sum_{F \leq x < P} \sum_{F \leq y < x} G(\Psi(y) \cdot \alpha_{\rho(y,x)}) \cdot \Psi([x,P]) \\
&= \sum_{F \leq y < P} \sum_{y < x < P} G(\Psi(y) \cdot \alpha_{\rho(y,x)}) \cdot \Psi([x,P]) \\
&= \sum_{F \leq y < P} G(\Psi(y)) \cdot \sum_{y < x < P} \alpha_{\rho(y,x)} \cdot \Psi([x,P]) \\
&\quad + \sum_{F \leq y < P} \Psi(y) \cdot \sum_{y < x < P} G(\alpha_{\rho(y,x)}) \cdot \Psi([x,P]) \\
&= \sum_{F \leq y < P} G(\Psi(y)) \cdot \big(\Psi([y,P]) - \beta_{\rho(y,P)-1}\big) \\
&\quad + \sum_{F \leq y < P} \Psi(y) \cdot \big(\alpha_{\rho(y,P)} - \mathrm{Pyr}(\beta_{\rho(y,P)-1})\big),
\end{aligned}$$

where the last step is by Proposition 6.2. Move the negative terms to the left-hand side and also cancel terms, we then obtain

$$\begin{aligned}
&\sum_{F \leq x < P} \big(G(\Psi(x)) \cdot \beta_{\rho(x,P)-1} + \Psi(x) \cdot \mathrm{Pyr}(\beta_{\rho(x,P)-1})\big) \\
&\geq \sum_{F \leq x < P} \Psi(x) \cdot \alpha_{\rho(x,P)}.
\end{aligned}$$

Now by the identity (2.1) we have

$$\operatorname{Pyr}\left(\sum_{F \leq x < P} \Psi(x) \cdot \beta_{\rho(x,P)-1}\right) \geq \sum_{F \leq x < P} \Psi(x) \cdot \alpha_{\rho(x,P)}.$$

Lastly, apply Corollaries 3.2 and 4.4 and the proof is complete.

Proposition 6.4 *Let P be a polytope with a vertex v. Let Q be the convex hull of P with a new vertex w placed beyond all facets of P containing v and beneath the rest. Then*

$$\Psi(P) \leq \Psi(Q).$$

Proof. Let $F_1, \ldots, F_r$ be facets of P that contain the vertex v and let $G_1, \ldots, G_s$ be the facets of Q that contain w. Then we have that $\partial(F_1 \cup \cdots \cup F_r) = \partial(G_1 \cup \cdots \cup G_s)$ (see [18, §5.2]). Moreover, the pyramid of $\partial(G_1 \cup \cdots \cup G_s)$ is $(G_1 \cup \cdots \cup G_s)'$.

Observe that the polytopes P and Q share the remaining facets. Let Λ be the union of these remaining facets. Now we have that

$$\begin{aligned}
\Psi(P) &= \Psi((F_1 \cup \cdots \cup F_r)') + \Psi(\Lambda') - \Psi(\partial(F_1 \cup \cdots \cup F_r)) \cdot \mathbf{c} \\
&\leq \operatorname{Pyr}(\Psi(\partial(F_1 \cup \cdots \cup F_r))) + \Psi(\Lambda') - \Psi(\partial(F_1 \cup \cdots \cup F_r)) \cdot \mathbf{c} \\
&= \Psi((G_1 \cup \cdots \cup G_s)') + \Psi(\Lambda') - \Psi(\partial(G_1 \cup \cdots \cup G_s)) \cdot \mathbf{c} \\
&= \Psi(Q),
\end{aligned}$$

where the inequality is Proposition 6.3 and the first and last step is Lemma 3.3.

Theorem 6.5 *Let P be a d-dimensional polytope with n vertices. Then*

$$\Psi(P) \leq \Psi(C(n,d)),$$

where $C(n,d)$ is the cyclic polytope in d dimensions and with n vertices.

Proof. We can pull each of the vertices of the polytope P until we obtain a simplicial polytope Q with the same number of vertices. By Proposition 6.4 we have that $\Psi(P) \leq \Psi(Q)$. Among simplicial polytopes McMullen [22, 23] proved that the cyclic polytope maximizes the h-vector. Since the **cd**-polynomials $\check{\Phi}_{d,i}$ are nonnegative, it follows that $\Psi(Q) \leq \Psi(C(n,d))$.

Since the flag h-vector is a nonnegative linear combination of the coefficients of the **cd**-index, and the flag f-vector a nonnegative linear combination of the flag h-vector, we obtain as a corollary two inequalities. The first of these was observed in [3, §7]; the second is new.

Corollary 6.6 *Let P be a d-dimensional polytope with n vertices. For $S \subseteq \{1, 2, \ldots, d\}$,*

$$f_S(P) \leq f_S(C(n,d)) \quad \textit{and} \quad h_S(P) \leq h_S(C(n,d)).$$

7 Derived inequalities

The strong submultiplicative inequalities for the $\mathbf{cd}$-index given in Theorem 5.1 give rise to quadratic inequalities for the flag f-vector. These arise by means of the convolution product on chain counts introduced in [20] and studied extensively in [11]. We compute the resulting derived inequalities for 4-dimensional polytopes.

First we determine the coefficients of the $\mathbf{cd}$-index in terms of the flag f-vector. To do this, we recall a related invariant, the sparse k-vector, defined in [9]. It was shown in [2] that a basis for the flag f-vectors of Eulerian posets of rank $d+1$ is given by the *sparse* subsets $S \subseteq \{1,\ldots,d\}$, namely those S not containing $\{i, i+1\}$, for all i, and not containing d. We define the *sparse flag k-vector*, for sparse subsets S, by

$$k_S = \sum_{T \subseteq S} (-1)^{|S|-|T|}\, h_T$$

or, equivalently,

$$k_S = \sum_{T \subseteq S} (-2)^{|S|-|T|}\, f_T. \tag{7.1}$$

It is immediate that the sparse k-vector also gives a basis for the flag f-vectors. This can be seen as well from the following unpublished result (due to the authors and Margaret Readdy), which can be proved by substituting the expression for k_S given in [9, Definition 6.5] into the expression below.

Proposition 7.1 *Let* $w = \mathbf{c}^{n_1}\mathbf{d}\mathbf{c}^{n_2}\mathbf{d}\mathbf{c}^{n_3}\cdots\mathbf{c}^{n_p}\mathbf{d}\mathbf{c}^{n_{p+1}}$ *be a* $\mathbf{cd}$*-word, and define* $m_0,\ldots,m_p$ *by* $m_0 = 1$ *and* $m_i = m_{i-1}+n_i+2$*. Then the coefficient of* w *in the* $\mathbf{cd}$*-index is given by*

$$\sum_{i_1,\ldots,i_p} (-1)^{(m_1-i_1)+(m_2-i_2)+\cdots+(m_p-i_p)}\, k_{i_1 i_2 \cdots i_p},$$

where the sum is over all p*-tuples* $(i_1, i_2, \ldots, i_p)$ *such that* $m_{j-1} \le i_j \le m_j - 2$*.*

To obtain these coefficients in terms of the f_S, one can invert the equation (7.1) to obtain

$$f_S = \sum_{T \subseteq S} 2^{|S|-|T|}\, k_T.$$

A different expression for the $\mathbf{cd}$-coefficients in terms of the flag h-vector is given in [4]. The $\mathbf{cd}$-coefficients were computed directly through dimension 8 (rank 9) by Meisinger [21, Appendix D]. The nonnegativity of the coefficients of the $\mathbf{cd}$-index for polytopes implies that the forms given in Proposition 7.1 are always nonnegative for polytopes. In fact, applying (5.1) when F is a facet of P shows that these forms are always increasing with respect to inclusion of faces. Since each k_S is itself a sum of **cd**-coefficients (see [9, pp. 34–35]), it follows as well that $k_S \geq 0$ for polytopes.

We can use the inequalities in Theorem 5.1 to obtain quadratic inequalities on the flag f-vector by making implicit use of a convolution on flag f-vectors studied in [11,20]. Suppose f_S^n and f_T^m count flags in posets of ranks m and n, respectively. Then if P is a poset of rank $n+m$, it follows that

$$\sum_{x:\rho(x)=n} f_S^n([\hat{0},x]) \cdot f_T^m([x,\hat{1}]) = f_{S\cup\{n\}\cup(T+n)}^{n+m}(P), \tag{7.2}$$

where $T+n := \{i+n \mid i \in T\}$. Thus if we sum an inequality like (5.1) or (5.2) over all faces F of a fixed dimension k, we get on the right-hand side a **cd**-polynomial with coefficients in terms of the flag f-vector of P (only involving flags containing k), while on the left-hand side we have the polynomial $f_k(P)\cdot\Psi(P)$, whose coefficients are products of linear forms in the flag f-vector with $f_k(P)$. We note that the convolution defined in (7.2) is dual to the coalgebra structure on $\mathbb{Z}\langle\mathbf{c},\mathbf{d}\rangle$ (see [11, Sect. 5.1]).

We illustrate by deriving all such inequalities that can be obtained for 4-dimensional polytopes. There are four cases to check, where the rank of F is 1, 2, 3 or 4 and the pyramid is performed on F. The other four cases will follow by duality. It follows from Proposition 7.1 that for posets of ranks between 1 and 5 the **cd**-indices are 1, $\mathbf{c}$, $\mathbf{c}^2+(f_1-2)\mathbf{d}$, $\mathbf{c}^3+(f_1-2)\mathbf{dc}+(f_2-f_1)\mathbf{cd}$ and $\mathbf{c}^4+(f_1-2)\mathbf{dc}^2+(f_2-f_1)\mathbf{cdc}+(f_3-f_2+f_1-2)\mathbf{c}^2\mathbf{d}+(f_{13}-2f_3-2f_1+4)\mathbf{d}^2$, respectively.

The table below summarizes the nontrivial inequalities coming from this approach. We have omitted those cases when the derived inequality is merely the nonnegativity of the relevant **cd**-coefficient. All inequalities are expressed in terms of the basis of sparse flag numbers f_1, f_2, f_3 and f_{13}.

$\rho(F)$	$\mathbf{cd}-$word	derived inequality
1	$\mathbf{c}^2\mathbf{d}$	$f_{13}+f_1f_2+2f_1 \leq f_1f_3+(f_1)^2+2f_2$
2	$\mathbf{c}^2\mathbf{d}$	$f_{13}+(f_2)^2 \leq f_1f_2+f_2f_3$
	$\mathbf{dc}^2$	$3 \leq f_1$
	$\mathbf{d}^2$	$2f_2(f_1+f_3) \leq f_{13}(f_2-1)+6f_2$
3	$\mathbf{dc}^2$	$f_{13}+f_3 \leq f_1f_3$
	$\mathbf{cdc}$	$f_{13}+f_1f_3 \leq f_3(f_2+1)$
4	$\mathbf{dc}^2$	$f_{13}+f_1f_2+3f_1+f_3 \leq f_1f_3+(f_1)^2+3f_2$
	$\mathbf{cdc}$	$f_{13}+f_1f_3+(f_1-f_2)^2+f_2 \leq f_2f_3+f_1+f_3$
	$\mathbf{c}^2\mathbf{d}$	$(f_1+f_3)(2f_2+3) \leq (f_1+f_3)^2+(f_2)^2+2f_1+f_2$
	$\mathbf{d}^2$	$f_{13}(f_2+1)+2(f_1+f_3)^2+6f_2 \leq (f_{13}+2f_2+6)(f_1+f_3)$

For completeness, we give below the inequalities derived in a similar way (for rational polytopes) from the Braden-MacPherson inequality (5.3). Here the relevant g-polynomials are 1, 1, $1 + (f_1 - 3)t$, $1 + (f_1 - 4)t$ and $1 + (f_1 - 5)t + (10 - 4f_1 + f_2 - 3f_3 + f_{13})t^2$ (see [21, Appendix B] or [4]). For these ranks, all the nontrivial inequalities come from comparing the degree one terms. Again, we omit those obtainable by duality.

$\rho(F)$	derived inequality
1	$2f_2 + f_1 \leq (f_1)^2$
2	$f_{13} + 2f_2 \leq f_1 f_2$
3	$f_{13} + 2f_3 \leq f_1 f_3$
4	$f_{13} + f_1 f_2 + 3f_1 + f_3 \leq f_1 f_3 + (f_1)^2 + 3f_2$

Note that the first of these is straightforward; in fact, it follows from a stronger inequality in [1, Theorem 2]. The fourth of these is exactly the coefficient of $\mathbf{dc}^2$ when $\rho(F) = 4$ in the first table.

It is interesting, and somewhat provocative, to note that for f-vectors of simplicial polytopes, the first nonlinear inequality of the characterizing set, $g_2 \leq g_1^{\langle 1 \rangle}$, is also quadratic. One can obtain further inequalities using this method by breaking P into more than two intervals and applying Theorem 5.1 or the Braden-MacPherson inequality successively. These will still be quadratic, with $f_S(P) \cdot \Psi(P)$ on the left-hand side and a polynomial with linear coefficients on the right-hand side. It is not clear whether or not the inequalities so obtained are consequences of the quadratics described above.

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References

1. Bayer, M.: The extended f-vectors of 4-polytopes. J. Combin. Theory Ser. A **44**, 141–151 (1987)
2. Bayer, M., Billera, L.: Generalized Dehn-Sommerville relations for polytopes, spheres and Eulerian partially ordered sets. Invent. Math. **79**, 143–157 (1985)
3. Bayer, M., Billera, L.: Counting faces and chains in polytopes and posets. In: Greene, C., ed.: Combinatorics and Algebra (Contemporary Mathematics, Vol. **34**, pp. 207–252) Providence: American Mathematical Society 1984
4. Bayer, M., Ehrenborg, R.: On the toric h-vectors of posets. To appear in Trans. Amer. Math. Soc.
5. Bayer, M., Klapper, A.: A new index for polytopes. Discrete Comput. Geom. **6**, 33–47 (1991)
6. Bayer, M., Lee, C.W.: Combinatorial Aspects of Convex Polytopes. In: Gruber, P.M., Wills, J.M., eds.: Handbook of Convex Geometry, Volume A (Chapter 2.3, pp. 485–534) Amsterdam: Elsevier Science Publishers B. V. 1993
7. Billera, L.J., Björner, A.: Face numbers of polytopes and complexes. In: Goodman, J.E., O'Rourke, J., eds.: Handbook of Discrete and Computational Geometry (pp. 291–310) Boca Raton New York: CRC Press 1997

8. Billera, L.J., Ehrenborg, R., Readdy, M.: The $\mathbf{c}$-$2\mathbf{d}$-index of oriented matroids. J. Combin. Theory Ser. A **80**, 79–105 (1997)
9. Billera, L.J., Ehrenborg, R., Readdy, M.: The **cd**-index of zonotopes and arrangements. In: Sagan, B.E., Stanley, R.P., eds.: Mathematical essays in honor of Gian-Carlo Rota (pp. 23–40) Boston: Birkhäuser 1998
10. Billera, L.J., Lee, C.W.: A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial polytopes. J. Combin. Theory Ser. A **31**, 237–255 (1981)
11. Billera, L.J., Liu, N.: Noncommutative enumeration in graded posets. To appear in J. Algebraic Combin.
12. Braden, T.C., MacPherson, R.: Intersection homology of toric varieties and a conjecture of Kalai. To appear in Comment. Math. Helv.
13. Bruggesser, H., Mani, P.: Shellable decompositions of spheres and cells. Math. Scand. **29**, 197–205 (1971)
14. Ehrenborg, R., Fox, H.: Products of posets and inequalities. Preprint 1999.
15. Ehrenborg, R., Hetyei, G.: Flags and shellings of Eulerian cubical posets. Preprint 1998.
16. Ehrenborg, R., Johnston, D., Rajagopalan, R., Readdy, M.: Cutting polytopes and flag f-vectors. To appear in Discrete Comput. Geom.
17. Ehrenborg, R., Readdy, M.: Coproducts and the $\mathbf{cd}$-index. J. Algebraic Combin. **8**, 273–299 (1998)
18. Grünbaum, B.: Convex Polytopes. New York: Wiley – Interscience 1967
19. Hetyei, G.: On the cd-variation polynomials of André and simsun permutations. Discrete Comput. Geom. **16**, 259–275 (1996)
20. Kalai, G.: A new basis of polytopes. J. Combin. Theory Ser. A **49**, 191–209 (1988)
21. Meisinger, G.: Flag Numbers and Quotients of Convex Polytopes. Doctoral dissertation, Universität Passau, 1993.
22. McMullen, P.: The maximum numbers of faces of a convex polytope. Mathematika **17**, 179–184 (1970)
23. McMullen, P., Shephard, G.C.: Convex polytopes and the upper bound conjecture. London: Cambridge University Press 1971
24. Montgomery, S.: Hopf Algebras and Their Actions on Rings (CBMS, Regional Conference Series in Mathematics, Number 82) Providence: American Mathematical Society 1993
25. Readdy, M.A.: Extremal problems for the Möbius function in the face lattice of the n-octahedron. Discrete Math., Special issue on Algebraic Combinatorics **139**, 361–380 (1995)
26. Stanley, R.P.: The number of faces of simplicial convex polytopes. Adv. Math. **35**, 236–238 (1980)
27. Stanley, R.P.: Enumerative Combinatorics, Vol. I. Pacific Grove: Wadsworth and Brooks/Cole 1986
28. Stanley, R.P.: Generalized h-vectors, intersection cohomology of toric varieties, and related results. In: Nagata, M., Matsumura, H., eds.: Commutative Algebra and Combinatorics (Advanced Studies in Pure Mathematics **11**, pp. 187–213) Tokyo: Kinokuniya and Amsterdam New York: North-Holland 1987
29. Stanley, R.P.: Flag f-vectors and the cd-index. Math. Z. **216**, 483–499 (1994)
30. Stanley, R.P.: A survey of Eulerian posets. In: Bisztriczky, T., McMullen, P., Schneider, R., Weiss, A.I., eds.: Polytopes: Abstract, Convex, and Computational (NATO ASI Series C, vol. 440, pp. 301–333) Dordrecht: Kluwer Academic Publishers 1994
31. Steinitz, E.: Über die Eulerischen Polyderrelationen. Arch. Math. Phys. **11**, 86–88 (1906)
32. Sweedler, M.: Hopf Algebras. New York: Benjamin 1969