# The **cd**-index of zonotopes and arrangements

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To Gian-Carlo Rota, for years of inspiration.

**Abstract**

We investigate a special class of polytopes, the zonotopes, and show that their flag $f$-vectors satisfy only the affine relations fulfilled by flag $f$-vectors of *all* polytopes. In addition, we determine the lattice spanned by flag $f$-vectors of zonotopes. By duality, these results apply as well to the flag $f$-vectors of central arrangements of hyperplanes.

## 1 Introduction

The flag $f$-vector of a convex polytope is an enumerative invariant of its lattice of faces, containing more information than the usual $f$-vector. While the latter counts the numbers of faces in each dimension, the former counts the numbers of chains (flags) having any possible set of dimensions.

The Euler relation is the only affine relation satisfied by $f$-vectors of all polytopes. For simplicial (or simple) $d$-polytopes, there are $\lfloor \frac{d-1}{2} \rfloor$ additional relations, called the Dehn-Sommerville equations, which provide a complete description of the affine space generated by all such $f$-vectors [10]. The information contained in the $f$-vector of a simplicial polytope is nicely summarized in the form of the $h$-vector [18].

In the case of the flag $f$-vector, there is a large set of equations that are satisfied for all polytopes. The corresponding affine space has dimension given by the Fibonacci sequence [1]. The **cd**-index provides an efficient way to summarize this information [2].

In the case of simplicial, simple and cubical polytopes, the flag $f$-vector reduces directly to the $f$-vector. In this paper we investigate another special class of polytopes, the zonotopes, and show for these that there is no

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reduction whatsoever; that is, we show that the flag $f$-vectors of zonotopes satisfy only the affine relations satisfied by flag $f$-vectors of *all* polytopes. This strengthens a result of Liu [14, Theorem 4.7.1]. Zonotopes are of particular interest in the study of hyperplane arrangements (see [20]), to which they are dual. A direct consequence of our result is that the **cd**-index of a central hyperplane arrangement is the most efficient encoding of the affine information of its flag $f$-vector.

We define the basic terminology used throughout this paper. For a convex $d$-dimensional polytope $Q$, and for a subset $S \subseteq \{0, \ldots, d-1\}$, we denote by $f_S$ the number of chains of faces (*flags*) in $Q$, $F_1 \subset \cdots \subset F_k$, with $S = \{\dim F_1, \ldots, \dim F_k\}$. The vector consisting of all the numbers $f_S$, $S \subseteq \{0, \ldots, d-1\}$, is called the *flag $f$-vector* of $Q$. The affine span of the flag $f$-vectors of all polytopes (more generally, of all Eulerian posets) is described by a system of linear equations, known as the generalized Dehn-Sommerville equations [1].

For any $S \subseteq \{0, \ldots, d-1\}$, we set $h_S = \sum_{T \subseteq S} (-1)^{|S \smallsetminus T|} f_T$. Define a polynomial in the non-commuting variables $\mathbf{a}$ and $\mathbf{b}$, called the **ab**-*index*, by

$$\Psi(Q) = \sum_S h_S \cdot u_S,$$

where $u_S = z_0 \cdots z_{d-1}$, $z_i = \mathbf{b}$ if $i \in S$ and $z_i = \mathbf{a}$ if $i \notin S$. An implicit encoding of the generalized Dehn-Sommerville equations is given by the fact that $\Psi(Q)$ is always a polynomial in the variables $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a}$. We call the polynomial the **cd**-*index* of $Q$.

As an example, the **cd**-index of a polygon $Q$ is given by

$$\Psi(Q) = \mathbf{c}^2 + (f_0 - 2) \cdot \mathbf{d} \tag{1.1}$$

and the **cd**-index of a 3-dimensional polytope $Q$ is given by

$$\Psi(Q) = \mathbf{c}^3 + (f_0 - 2) \cdot \mathbf{dc} + (f_2 - 2) \cdot \mathbf{cd}. \tag{1.2}$$

In Section 2, we discuss the operations of taking pyramids and prisms, and we use them to give a direct proof that the flag $f$-vectors of all polytopes span the linear space determined by the generalized Dehn-Sommerville equations. We next discuss zonotopes and three operations on them – projection, Minkowski sum with a line segment and prism. We use the coalgebra techniques of [7] to determine their effect on the **cd**-index. In Section 4, we show the **cd**-index of an $n$-fold iterated Minkowski sum is a polynomial function of $n$, and we use this in Section 5 to show that the flag $f$-vectors of zonotopes also span the space of all flag $f$-vectors. This result is extended in Section 6 by determining the lattice spanned by the **cd**-indices of all zonotopes. It is the ring of all integral polynomials in $\mathbf{c}$

and $2\mathbf{d}$. In terms of flag $f$-vectors, this is equivalent to saying that $f_S$ is divisible by $2^{|S|}$. Some observations and concluding remarks are indicated in the final section.

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## 2 Polytopes span

For a field $\mathbf{k}$ of characteristic 0, let $\mathcal{F}$ be the polynomial algebra in non-commuting variables $\mathbf{c}$ and $\mathbf{d}$ over the field $\mathbf{k}$, that is, $\mathcal{F} = \mathbf{k}\langle \mathbf{c}, \mathbf{d} \rangle$. (In fact, everything we do here works in any characteristic other than 2.) If we set the degree of $\mathbf{c}$ to 1 and the degree of $\mathbf{d}$ to 2, we define $\mathcal{F}_d$ to be all polynomials in $\mathcal{F}$ that are homogeneous of degree $d$.

Recall that a derivation $f$ on an algebra $A$ is a linear map satisfying the product rule $f(x \cdot y) = f(x) \cdot y + x \cdot f(y)$, and that $f(1) = 0$. Observe that it is enough to determine how the derivation acts on a set of generators, and hence we may describe a derivation on $\mathcal{F}$ by giving its value on the elements $\mathbf{c}$ and $\mathbf{d}$. Define two derivations $D$ and $G$ on $\mathcal{F}$ by $D(\mathbf{c}) = 2 \cdot \mathbf{d}$, $D(\mathbf{d}) = \mathbf{c} \cdot \mathbf{d} + \mathbf{d} \cdot \mathbf{c}$, $G(\mathbf{c}) = \mathbf{d}$, and $G(\mathbf{d}) = \mathbf{c} \cdot \mathbf{d}$. Observe that both these derivations increase the degree by 1, that is, they are maps from $\mathcal{F}_d$ to $\mathcal{F}_{d+1}$.

For a polytope $Q$ we denote the pyramid over $Q$ by $\mathrm{Pyr}(Q)$. Likewise, denote the prism over $Q$ by $\mathrm{Pri}(Q)$. We similarly denote two linear maps $\mathrm{Pyr}, \mathrm{Pri} : \mathcal{F} \to \mathcal{F}$, by

$$\mathrm{Pyr}(w) = w \cdot \mathbf{c} + G(w)$$

and

$$\mathrm{Pri}(w) = w \cdot \mathbf{c} + D(w).$$

The following results are proved by in [7] using coalgebra techniques (see [7, Theorems 4.4 and 5.2]).

**Proposition 2.1** *For a polytope $Q$ we have that*

$$\begin{aligned} \Psi(\mathrm{Pyr}(Q)) &= \mathrm{Pyr}(\Psi(Q)), \\ \Psi(\mathrm{Pri}(Q)) &= \mathrm{Pri}(\Psi(Q)). \end{aligned}$$

**Lemma 2.2** *The linear span of the two sets* $\mathrm{Pyr}(\mathcal{F}_d)$ *and* $\mathrm{Pri}(\mathcal{F}_d)$ *is the linear space* $\mathcal{F}_{d+1}$.

**Proof:** Define a third derivation $G'$ on the algebra $\mathcal{F}$ by $G'(\mathbf{c}) = \mathbf{d}$ and $G'(\mathbf{d}) = \mathbf{d} \cdot \mathbf{c}$. It follows that $w \cdot \mathbf{c} + G(w) = \mathbf{c} \cdot w + G'(w)$ for all $w \in \mathcal{F}$ (see [7, Lemma 5.1]).

Observe that $\mathrm{Pri}(w) - \mathrm{Pyr}(w) = D(w) - G(w) = G'(w)$. Thus the statement of the lemma is equivalent to that $\mathrm{Pyr}(\mathcal{F}_d)$ and $G'(\mathcal{F}_d)$ span the space $\mathcal{F}_{d+1}$. Let $V = \mathrm{Pyr}(\mathcal{F}_d) + G'(\mathcal{F}_d)$.

Let $w$ be an element in $\mathcal{F}_d$. Then we have that $\mathbf{c} \cdot w = w \cdot \mathbf{c} + G(w) - G'(w) = \mathrm{Pyr}(w) - G'(w)$. Hence $\mathbf{c} \cdot w$ belongs to $V$.

Let $v$ be in $\mathcal{F}_{d-1}$. Then we have that $G'(\mathbf{c} \cdot v) = \mathbf{d} \cdot v + \mathbf{c} \cdot G'(v)$. Since $\mathbf{c} \cdot G'(v)$ belongs to $V$ by the previous paragraph and $G'(\mathbf{c} \cdot v)$ also belongs to $V$, we have $\mathbf{d} \cdot v \in V$.

Since a monomial in $\mathcal{F}_{d+1}$ begins either with a $\mathbf{c}$ or a $\mathbf{d}$, we conclude $V = \mathcal{F}_{d+1}$. ■

From Lemma 2.2, we conclude directly the basic result that the linear span of all flag $f$-vectors has dimension given by the Fibonacci numbers [1].

**Theorem 2.3** *Beginning with a point one can produce, by repeated use of the operations* Pyr *and* Pri*, a set of polytopes whose* **cd***-indices span* $\mathcal{F}$*.*

We note that this approach does not identify a specific basis, as was done in [1] and [12]. We end this section with a few useful facts.

**Lemma 2.4** *The two linear maps* Pri *and* Pyr *are injective. The linear map $G$ has kernel generated by* 1*, and the linear map $D$ has kernel generated by the elements of the form* $(\mathbf{c}^2 - 2 \cdot \mathbf{d})^j$*,* $j \geq 1$*.*

**Proof:** Let $\mathcal{F}_d^{(i)}$ be the linear span of all monomials of degree $d$ containing $i$ $\mathbf{d}$'s. Define two derivations $D_0$ and $D_1$ by: $D_0(\mathbf{c}) = 0$, $D_0(\mathbf{d}) = \mathbf{cd} + \mathbf{dc}$, $D_1(\mathbf{c}) = 2 \cdot \mathbf{d}$, and $D_1(\mathbf{d}) = 0$. Define two linear maps $\mathrm{Pri}_0$ and $\mathrm{Pri}_1$ by: $\mathrm{Pri}_0(v) = D_0(v) + v \cdot \mathbf{c}$ and $\mathrm{Pri}_1(v) = D_1(v)$. We have that $\mathrm{Pri} = \mathrm{Pri}_0 + \mathrm{Pri}_1$, and $\mathrm{Pri}_j$ is a linear map from $\mathcal{F}_d^{(i)}$ to $\mathcal{F}_{d+1}^{(i+j)}$.

Define a linear map $\phi : \ \mathcal{F}_d^{(i)} \rightarrow \mathbf{k}[x_0, \ldots, x_i]$ by

$$\phi(\mathbf{c}^{n_0}\mathbf{d}\mathbf{c}^{n_1}\mathbf{d} \cdots \mathbf{d}\mathbf{c}^{n_i}) = x_0^{n_0} x_1^{n_1} \cdots x_i^{n_i}.$$

This map takes the linear space $\mathcal{F}_d^{(i)}$ isomorphically onto the linear space of homogeneous polynomials of degree $d - 2 \cdot i$ in the variables $x_0, \ldots, x_i$. Moreover, we have

$$\phi(\mathrm{Pri}_0(w)) = (x_0 + 2 \cdot x_1 + \cdots + 2 \cdot x_i) \cdot \phi(w).$$

Since the ring of polynomials is an integral domain, we have $(x_0 + 2 \cdot x_1 + \cdots + 2 \cdot x_i)$ is not a zero divisor. Hence $\mathrm{Pri}_0 : \mathcal{F}_d^{(i)} \longrightarrow \mathcal{F}_{d+1}^{(i)}$ is an injective map. The linear map Pri corresponds to a block matrix $B$, where the row labels of the block $B_{i,j}$ correspond to monomials having $i$ $\mathbf{d}$'s and the column labels correspond to monomials having $j$ $\mathbf{d}$'s. The blocks on the diagonal of $B$ are described by $\mathrm{Pri}_0$ and the blocks below the diagonal are equal to zero. We thus conclude Pri is also injective.

The proof is similar for the linear map Pyr. The only difference is that we obtain another polynomial of degree 1, namely the polynomial $x_0 + \cdots + x_i$.

For the two linear maps $D$ and $G$ we need to modify the argument. For $D$ we get the polynomial $x_0 + 2x_1 + \cdots + 2x_{i-1} + x_i$ and for $G$ we get $x_0 + \cdots + x_{i-1}$. We can now obtain that the two linear maps $D_0, G_0 : \mathcal{F}_d^{(i)} \longrightarrow \mathcal{F}_{d+1}^{(i)}$ are injective when $i \geq 1$, while when $i = 0$ they are the zero maps. Since $\dim(\mathcal{F}_d^{(0)}) = 1$, the dimensions of the kernels of the linear maps $D, G : \mathcal{F}_d \longrightarrow \mathcal{F}_{d+1}$ are each at most 1.

For $d \geq 1$ it is easy to see that $G$ restricted to $\mathcal{F}_d^{(0)} \oplus \mathcal{F}_d^{(1)}$ is an injective map. Thus $G$ can still be divided into blocks so that the blocks on the main diagonal are injective. Hence we conclude that for $d \geq 1$ the map $G : \mathcal{F}_d \longrightarrow \mathcal{F}_{d+1}$ is injective, that is, the kernel of $G$ is generated by the polynomial 1.

When $d$ is odd one can similarly obtain that $D$ restricted to $\mathcal{F}_d^{(0)} \oplus \mathcal{F}_d^{(1)}$ is an injective map. Hence $D : \mathcal{F}_d \longrightarrow \mathcal{F}_{d+1}$ is an injective map for $d$ odd. Finally, when $d$ is even it is easy to see that $D\left((\mathbf{c}^2 - 2 \cdot \mathbf{d})^{d/2}\right) = 0$. Hence the kernel of $D$ is generated by elements of the form $(\mathbf{c}^2 - 2 \cdot \mathbf{d})^j$. ■

**Corollary 2.5** *For all non-negative integers $k$ we have that $D^k(\mathbf{c})$ is non-zero.*

**Proof:** The proof is by induction on $k$. It follows directly for $k = 0$ and $k = 1$. Assume for $k \geq 1$ that $D^k(\mathbf{c})$ is non-zero. Observe that the coefficient of $\mathbf{c}^{k+1}$ in $D^k(\mathbf{c})$ is zero. Hence $D^k(\mathbf{c})$ is not a scalar multiple of $(\mathbf{c}^2 - 2 \cdot \mathbf{d})^j$ and so $D^k(\mathbf{c})$ does not belong to the kernel of $D$. We conclude that $D^{k+1}(\mathbf{c})$ is non-zero. ■

# 3 Zonotopes

The *Minkowski sum* of two subsets $X$ and $Y$ of $\mathbb{R}^d$ is defined as

$$X + Y = \{\mathbf{x} + \mathbf{y} \in \mathbb{R}^d \ : \ \mathbf{x} \in X, \mathbf{y} \in Y\}.$$

Notably, the Minkowski sum of two convex polytopes is another convex polytope. For a vector $\mathbf{x}$ we denote the set $\{\lambda \cdot \mathbf{x} \ : \ 0 \leq \lambda \leq 1\}$ by $[\mathbf{0}, \mathbf{x}]$.

We denote by $\text{aff}(X)$ the *affine span* of $X$, that is, the intersection of all affine subspaces containing the set $X$.

We say that the nonzero vector $\mathbf{x} \in \text{aff}(Q)$ lies in *general position* with respect to the convex polytope $Q$ if the line $\{\lambda \cdot \mathbf{x} + \mathbf{u} \in \mathbb{R}^d \ : \ \lambda \in \mathbb{R}\}$ intersects the boundary of the polytope $Q$ in at most two points for all $\mathbf{u} \in \mathbb{R}^d$. Alternatively, $\mathbf{x} \in \text{aff}(Q)$ is in general position if $\mathbf{x}$ is parallel to no proper face of $Q$.

From [7, Prop. 6.3] we have the following result. Let $Q$ be a $d$-dimensional convex polytope and $\mathbf{x}$ a nonzero vector that lies in general position with respect to the polytope $Q$. Let $H$ be a hyperplane orthogonal to the vector $\mathbf{x}$, and let $\text{Proj}(Q)$ be the orthogonal projection of $Q$ onto the hyperplane $H$. Observe that $\text{Proj}(Q)$ is a $(d-1)$-dimensional convex polytope.

**Proposition 3.1** *The* **cd***-index of the Minkowski sum of $Q$ and $[\mathbf{0}, \mathbf{x}]$ is given by*

$$\Psi(Q + [\mathbf{0}, \mathbf{x}]) = \Psi(Q) + D(\Psi(\text{Proj}(Q))).$$

A *zonotope* is the Minkowski sum of line segments. That is, if $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^d$, then the zonotope they generate is the Minkowski sum

$$Z = [\mathbf{0}, \mathbf{x}_1] + \cdots + [\mathbf{0}, \mathbf{x}_n].$$

A (central) *hyperplane arrangement* is a finite collection $\mathcal{H}$ of linear hyperplanes in $\mathbb{R}^d$. An arrangement is called *essential* if the intersection of all its hyperplanes is the origin. An arrangement $\mathcal{H}$ induces a subdivision of $\mathbb{R}^d$ into relatively open cells whose closures are ordered by inclusion. The resulting poset is a lattice, called the *face lattice* of $\mathcal{H}$. An arrangement $\mathcal{H} \subset \mathbb{R}^d$ has a natural flag $f$-vector with components $f_S(\mathcal{H})$, where $S \subseteq \{1, \ldots, d\}$. The face lattice of $Z$ is anti-isomorphic to that of the central arrangement $\mathcal{H}$ of the $n$ hyperplanes with normals $\mathbf{x}_1, \ldots, \mathbf{x}_n$ [5, Prop. 2.2.2]. If $Z$ is $d$-dimensional, then its flag $f$-vector and that of its dual hyperplane arrangement are related by $f_S(Z) = f_{d-S}(\mathcal{H})$, where $S = \{i_1, \ldots, i_k\} \subseteq \{0, \ldots, d-1\}$ and $d - S = \{d - i_k, \ldots, d - i_1\}$.

Two important and useful facts about the combinatorial behavior of zonotopes are the following:

1. The face lattice of $Z$ is determined by the *oriented matroid* of the point configuration $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ [5, Prop. 2.2.2], and

2. The flag $f$-vector of $Z$ is determined by the *matroid* of the configuration $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ [5, Cor. 4.6.3].

Since we are only interested here in invariants derivable from the flag $f$-vector, we will consider two zonotopes to be equal if they have the same underlying matroid. In this case it will be important, when defining operations on zonotopes, to show that they only depend on their matroids.

For a zonotope $Z$ we note that the combinatorial type of the prism over $Z$ can be realized as the zonotope $\mathrm{Pri}(Z) = Z + [\mathbf{0}, \mathbf{x}]$ for any $\mathbf{x} \notin \mathrm{aff}(Z)$. At the level of matroids, this involves adding a new element independent of all the original ones, that is, forming a free extension of one higher rank.

We define a zonotope $M(Z)$ by

$$M(Z) = Z + [\mathbf{0}, \mathbf{x}],$$

where $\mathbf{x}$ lies in general position with respect to $Z$. While the combinatorial type of $M(Z)$ depends on the choice of $\mathbf{x}$, its matroid is well-defined. This follows since the underlying matroid of $M(Z)$ is always a free extension (of the same rank) of the matroid of $Z$, that is, an extension such that $\mathbf{x}$ lies on no proper subspace spanned by the generators $\mathbf{x}_1, \ldots, \mathbf{x}_n$.

Finally, we define the zonotope $\pi(Z)$ to be the projection of $M(Z)$ along the direction $\mathbf{x}$, that is, onto the hyperplane orthogonal to $\mathbf{x}$. Observe that $\pi(Z)$ is the projection $\mathrm{Proj}(Z)$ in a general direction. The matroid of the zonotope $\pi(Z)$ is well-defined, since it is obtained by contracting $\mathbf{x}$ in the matroid of $M(Z)$.

Directly as a corollary of Proposition 3.1 we have

**Corollary 3.2** *For a zonotope $Z$ we have*

$$\Psi(M(Z)) - \Psi(Z) = D(\Psi(\pi(Z))).$$

The operations Pri, $M$, and $\pi$ were used by Liu [14] to give a lower bound on the dimension of the span of the flag $f$-vectors of zonotopes. The relationship between these operations is given by the following lemma. The second relation was first observed by Liu in [14, Theorem 4.2.7].

**Lemma 3.3** *For a zonotope $Z$ we have, up to matroid,*

$$\pi(M(Z)) = M(\pi(Z))$$

*and*

$$\pi(\mathrm{Pri}(Z)) = M(Z).$$

**Proof:** In each pair we check that the underlying matroids are the same.

For $\pi(M(Z))$ one makes a free extension of $Z$ by $\mathbf{x}$ and again by $\mathbf{y}$ (both in $\text{aff}(Z)$), then contracting $\mathbf{y}$. The image of $\mathbf{x}$ under this contraction is still free with respect to the images of $\mathbf{x}_1, \ldots, \mathbf{x}_n$, so the resulting matroid is the same as that of $M(\pi(Z))$.

For $\pi(\text{Pri}(Z))$ the description is the same, except now neither $\mathbf{x}$ nor $\mathbf{y}$ is in $\text{aff}(Z)$. In this case, the images of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ will have the same matroid as $M(Z)$. ■

## 4 Polynomial functions

In this section we define polynomial functions and derive some of their properties. These functions play a role in the proof of our main theorem. Let $V$ and $W$ be vector spaces over the field $\mathbf{k}$.

**Definition 4.1** *A function $f : \mathbb{N} \longrightarrow V$ is called a* polynomial function of degree $d$ *if it can be written in the form*

$$f(n) = \mathbf{v}_d \cdot \binom{n}{d} + \mathbf{v}_{d-1} \cdot \binom{n}{d-1} + \cdots + \mathbf{v}_0 \cdot \binom{n}{0},$$

*where $\mathbf{v}_0, \ldots, \mathbf{v}_d \in V$ and $\mathbf{v}_d \neq \mathbf{0}$. We call $\mathbf{v}_d$ the* leading coefficient*.*

Observe that $\binom{n}{d}$ is defined by the Pascal relations in any characteristic. We define the *difference operator* $\Delta$ by $\Delta f(n) = f(n+1) - f(n)$. The following proposition contains the essential results we will need about polynomial functions.

**Proposition 4.2** *Let $f : \mathbb{N} \longrightarrow V$ be a polynomial function of degree $d$.*

*(i) If $\phi : V \longrightarrow W$ is a linear map then the composition $\phi \circ f : \mathbb{N} \longrightarrow W$ is a polynomial function of degree at most $d$. If $\phi$ applied to the leading coefficient is non-zero then the degree is $d$.*

*(ii) The function $\Delta f(n)$ is a polynomial function of degree $d - 1$.*

*(iii) If $g$ is a function from $\mathbb{N}$ to $V$ such that $\Delta g(n) = f(n)$ then $g$ is a polynomial function of degree $d + 1$ with the same leading coefficient as $f$.*

*(iv) The vector $f(0)$ is in the linear span of $f(1), \ldots, f(d + 1)$.*

**Proof:** Let $f(n)$ be the polynomial function of degree $d$

$$f(n) = \mathbf{v}_d \cdot \binom{n}{d} + \mathbf{v}_{d-1} \cdot \binom{n}{d-1} + \cdots + \mathbf{v}_0 \cdot \binom{n}{0}.$$

(i) Observe that

$$(\phi \circ f)(n) = \phi(\mathbf{v}_d) \cdot \binom{n}{d} + \phi(\mathbf{v}_{d-1}) \cdot \binom{n}{d-1} + \cdots + \phi(\mathbf{v}_0) \cdot \binom{n}{0},$$

which is a polynomial function of degree at most $d$. When $\phi(\mathbf{v}_d) \neq \mathbf{0}$ we have that $\phi \circ f$ is of degree $d$.

(ii) It is straightforward to obtain

$$\Delta f(n) = \mathbf{v}_d \cdot \binom{n}{d-1} + \mathbf{v}_{d-1} \cdot \binom{n}{d-2} + \cdots + \mathbf{v}_1 \cdot \binom{n}{0},$$

which proves (ii).

(iii) By induction on $n$ we have

$$g(n) = \mathbf{v}_d \cdot \binom{n}{d+1} + \mathbf{v}_{d-1} \cdot \binom{n}{d} + \cdots + \mathbf{v}_0 \cdot \binom{n}{1} + g(0),$$

which is a polynomial function of degree $d+1$. The leading coefficient is $\mathbf{v}_d$, which is the leading coefficient of $f$.

(iv) By property (ii) we know that $\Delta^d f(n)$ is a polynomial function of degree 0, hence it is a constant. Thus $\Delta^d f(0) = \Delta^d f(1)$. But $\Delta^d f(0)$ is a linear combination of $f(0), \ldots, f(d)$ and $\Delta^d f(1)$ is a linear combination of $f(1), \ldots, f(d+1)$. The coefficient of $f(0)$ in $\Delta^d f(0)$ is $(-1)^d$, which is nonzero, and hence the relation $\Delta^d f(0) = \Delta^d f(1)$ gives the desired result. ■

Observe that Proposition 4.2 and its proof are valid in any characteristic for the field $\mathbf{k}$ since $(-1)^d$ is never zero. Moreover, it applies to Abelian groups ($\mathbb{Z}$-modules) as well. This last fact will be used in Section 6.

The main result of this section shows that the **cd**-index of iterates of the operation $M$ is a polynomial function.

**Theorem 4.3** *Let $Z$ be a $d$-dimensional zonotope. Then the mapping $n \mapsto \Psi(M^n(Z))$ is a polynomial function of degree $d-1$ into $\mathcal{F}_d$ with leading coefficient $D^{d-1}(\mathbf{c})$.*

**Proof:** The proof is by induction on $d$. The base case is $d = 2$. Assume that $Z$ is a 2-dimensional zonotope, that is, $Z$ is a $2k$-gon. Then $M(Z)$ is a $(2k+2)$-gon, and $M^n(Z)$ is a $(2k+2n)$-gon. By equation (1.1) we have the **cd**-index of $M^n(Z)$ is given by $\Psi(M^n(Z)) = \mathbf{c}^2 + (2k+2n-2)\cdot\mathbf{d} = 2\cdot n\cdot\mathbf{d} + \mathbf{c}^2 + (2k-2)\cdot\mathbf{d}$. This is a polynomial function of degree 1 in $n$ with leading coefficient $2\cdot\mathbf{d} = D(\mathbf{c})$.

Assume that $d \geq 3$ and let $W = \pi(Z)$. Observe that $W$ is a $(d-1)$-dimensional zonotope. Now by Corollary 3.2 and Lemma 3.3 we have

$$\begin{aligned}\Delta(\Psi(M^n(Z))) &= \Psi(M^{n+1}(Z)) - \Psi(M^n(Z)) \\ &= D(\Psi(M^n(\pi(Z)))) \\ &= D(\Psi(M^n(W))).\end{aligned}$$

By the induction hypothesis we know that $n \mapsto \Psi(M^n(W))$ is a polynomial function of degree $d-2$ with leading coefficient $D^{d-2}(\mathbf{c})$. By Corollary 2.5 and by property (i) in Proposition 4.2, we have $n \mapsto D(\Psi(M^n(W)))$ is polynomial function of degree $d-2$ with non-zero leading term $D^{d-1}(\mathbf{c})$. Finally, by property (iii) in the same proposition we complete the induction. ■

# 5 Zonotopes span

Let $\mathcal{G}_d$ be the linear span of the **cd**-indices of zonotopes of dimension $d$. Liu proved that $\dim \mathcal{G}_d \geq \dim \mathcal{G}_{d-1} + \dim \mathcal{G}_{d-3}$ [14, Theorem 4.7.1]. In this section we prove that $\dim \mathcal{G}_d = \dim \mathcal{G}_{d-1} + \dim \mathcal{G}_{d-2}$, that is, $\mathcal{G}_d$ equals $\mathcal{F}_d$.

Since zonotopes are polytopes, we know that $\mathcal{G}_d \subseteq \mathcal{F}_d$. We first prove a variation of Lemma 2.2 that substitutes $D$ for Pyr in order to be able to operate solely with zonotopes.

**Lemma 5.1** *The linear span of the two sets $D(\mathcal{F}_d)$ and $\mathrm{Pri}(\mathcal{F}_d)$ is the whole space $\mathcal{F}_{d+1}$.*

**Proof:** Let $V$ be the subspace of $\mathcal{F}_d$ which is spanned by $D(\mathcal{F}_d)$ and $\mathrm{Pri}(\mathcal{F}_d)$, that is, $V = D(\mathcal{F}_d) + \mathrm{Pri}(\mathcal{F}_d)$.

Let $w \in \mathcal{F}_d$. Since $w\cdot\mathbf{c} = \mathrm{Pri}(w) - D(w) \in V$, we know that every **cd**-monomial which ends with a $\mathbf{c}$ belongs to $\mathrm{Pri}(\mathcal{F}_d) + D(\mathcal{F}_d)$.

Consider $v \in \mathcal{F}_{d-1}$. We have that $D(v\cdot\mathbf{c}) = D(v)\cdot\mathbf{c} + 2\cdot v\cdot\mathbf{d}$, and hence $v\cdot\mathbf{d} = \frac{1}{2}\cdot(D(v\cdot\mathbf{c}) - D(v)\cdot\mathbf{c})$. We have $D(v\cdot\mathbf{c}) \in V$. Moreover $D(v)\cdot\mathbf{c} \in V$ by the previous paragraph. Hence $v\cdot\mathbf{d} \in V$, and we conclude that every **cd**-monomial belongs to $V$. ■

The following result shows that the flag $f$-vectors of zonotopes made by the successive application of the operators Pri and $M$, beginning with $Z = \mathbf{0}$, span the space of all flag $f$-vectors of polytopes.

**Theorem 5.2** *The **cd**-indices of $d$-dimensional zonotopes linearly span the space of **cd**-polynomials of degree $d$, that is, $\mathcal{G}_d = \mathcal{F}_d$.*

**Proof:** The proof is by induction on the dimension $d$; the case $d \leq 2$ is clear. We assume that the theorem holds for $d \geq 2$, hence $\mathcal{G}_d = \mathcal{F}_d$, and prove it for $d+1$. Assume that $\{Z_1, \ldots, Z_N\}$ form a spanning set of zonotopes of dimension $d$. Since $\Psi(\mathrm{Pri}(Z_i)) = \mathrm{Pri}(\Psi(Z_i))$ we have that $\mathrm{Pri}(\mathcal{F}_d) \subseteq \mathcal{G}_{d+1}$.

By combining Theorem 4.3 and property (iv) in Proposition 4.2, we know that $\Psi(Z_i)$ lies in the linear span of $\Psi(M(Z_i)), \ldots, \Psi(M^d(Z_i))$. Hence, we know that $\left\{M^j(Z_i) \mid 1 \leq i \leq N, 1 \leq j \leq d\right\}$ is a spanning set of zonotopes. Observe that every zonotope in this spanning set is the Minkowski sum of a line segment with a $d$-dimensional zonotope. Hence we can describe this spanning set as $\{M(W_1), \ldots, M(W_{N \cdot d})\}$.

By Lemma 3.3 and Corollary 3.2 we have

$$\begin{aligned}\Psi(M(\mathrm{Pri}(W_i))) - \Psi(\mathrm{Pri}(W_i)) &= D(\Psi(\pi(\mathrm{Pri}(W_i)))) \\ &= D(\Psi(M(W_i))).\end{aligned}$$

Since both $M(\mathrm{Pri}(W_i))$ and $\mathrm{Pri}(W_i)$ are $(d+1)$-dimensional zonotopes, we have $D(\Psi(M(W_i))) \in \mathcal{G}_{d+1}$. But since $\{M(W_i)\}$ forms a spanning set for $\mathcal{F}_d$, we obtain that $D(\mathcal{F}_d) \subseteq \mathcal{G}_{d+1}$. By Lemma 5.1 we obtain that $\mathcal{G}_{d+1} = \mathcal{F}_{d+1}$, which completes the induction. ■

Since the face lattice of a central hyperplane arrangement is an Eulerian poset, it has a **cd**-index, obtainable from that of its dual zonotope by reversing each **cd**-monomial.

**Corollary 5.3** *The **cd**-indices of essential hyperplane arrangements in $\mathbb{R}^d$ linearly span the space of **cd**-polynomials of degree $d$.*

# 6 The integral span

We turn now to the problem of finding the *integral* span of flag $f$-vectors of zonotopes. This leads to an integral **c**-2**d**–index for zonotopes and central arrangements.

Let $\mathcal{R}$ be the ring in the non-commuting variables **c** and **d** over the integers $\mathbb{Z}$, that is, $\mathcal{R} = \mathbb{Z}\langle \mathbf{c}, \mathbf{d} \rangle$. As before let the degree of **c** be 1 and the degree of **d** be 2. Let $\mathcal{R}_d$ be all polynomials in $\mathcal{R}$ that are homogeneous of degree $d$. We view $\mathcal{R}_d$ as an Abelian group. Similarly, let $\mathcal{T} = \mathbb{Z}\langle \mathbf{c}, 2\mathbf{d} \rangle$ and let $\mathcal{T}_d = \mathcal{T} \cap \mathcal{R}_d$. For a **cd**-monomial $w$, let $p(w)$ be the number of **d**'s that occur in $w$. A generating set of $\mathcal{T}_d$ is $2^{p(w)} \cdot w$, where $w$ ranges over all **cd**-monomials of degree $d$.

Observe that Lemma 2.2 and Theorem 2.3 have the following integer analogues.

**Lemma 6.1** *The Abelian group $\mathcal{R}_{d+1}$ is generated by* $\text{Pyr}(\mathcal{R}_d)$ *and* $\text{Pri}(\mathcal{R}_d)$.

**Theorem 6.2** *The Abelian group $\mathcal{R}_d$ is generated by the* **cd***-index of $d$-dimensional polytopes.*

The goal of this section is to prove the analogous result of Theorem 6.2 for zonotopes. Let $\mathcal{S}_d$ be the subgroup of $\mathcal{R}_d$ generated by the elements $\Psi(Z)$, where $Z$ ranges over all $d$-dimensional zonotopes. We begin by showing that $\mathcal{T}_d \subseteq \mathcal{S}_d$. This proof is essentially the same as the proof of Theorem 5.2. We need the following lemma.

**Lemma 6.3** *The Abelian group $\mathcal{T}_{d+1}$ is generated by* $\text{Pri}(\mathcal{T}_d)$ *and* $D(\mathcal{T}_d)$.

The proof differs from the proof of Lemma 5.1 in only one point. We do not divide by 2; we instead use the relation $2 \cdot v \cdot \mathbf{d} = D(v \cdot \mathbf{c}) - D(v) \cdot \mathbf{c}$ and the fact that the monomial $v \cdot \mathbf{d}$ contains one more $\mathbf{d}$ than $v$, that is, $p(v \cdot d) = p(v) + 1$. We thus have that the generating set of $\mathcal{T}_{d+1}$ lies in the integral span of $\text{Pri}(\mathcal{T}_d)$ and $D(\mathcal{T}_d)$.

The results in Section 4 also apply to Abelian groups as well as vector spaces. Hence the proof of Theorem 5.2 generalizes to a proof of the following result.

**Proposition 6.4** *The Abelian group $\mathcal{T}_d$ is contained in the group $\mathcal{S}_d$.*

It remains to show the inclusion in the other direction, that is, $\mathcal{S}_d \subseteq \mathcal{T}_d$. For $S$ a subset of $\{0, 1, \ldots, d-1\}$, we call $S$ *sparse* if for all $i$, $\{i, i+1\} \not\subseteq S$ and $d-1 \notin S$. Suppose that $S$ has cardinality $p$. Let $w$ be a **cd**-monomial of degree $d$ containing $p$ **d**'s. We say that $w$ *covers* the sparse set $S$ if $u_S$ appear in the expansion of $w = w(\mathbf{c}, \mathbf{d})$ as an **ab**-polynomial $w = w(\mathbf{a}+\mathbf{b}, \mathbf{ab}+\mathbf{ba})$. More explicitly, we can write $w = \mathbf{c}^{i_0} \cdot \mathbf{d} \cdot \mathbf{c}^{i_1} \cdot \mathbf{d} \cdots \mathbf{d} \cdot \mathbf{c}^{i_p}$, where $i_k \geq 0$. Define $j_0, \ldots, j_{p-1}$ by $j_0 = i_0$ and $j_{h+1} = j_h + 2 + i_{h+1}$. Observe that the $h$th $\mathbf{d}$ in $w$ covers the positions $j_h$ and $j_{h+1}$. Then $w$ covers the sparse set $S$ if and only if $S$ is contained in the set $\{j_0, j_0+1, j_1, j_1+1, \ldots, j_{p-1}, j_{p-1}+1\}$. (Compare this notion with Stanley's definition of $\mathcal{W}_S$ [16].)

For a **cd**-monomial $w$ and a **cd**-polynomial $F(\mathbf{c}, \mathbf{d})$, we denote the coefficient of $w$ in $F(\mathbf{c}, \mathbf{d})$ by $[w]F(\mathbf{c}, \mathbf{d})$.

**Definition 6.5** *For a $d$-dimensional polytope $Q$ and a sparse subset $S$ of $\{0,1,\ldots,d-1\}$, define $k_S$ by*

$$k_S = \sum_w [w]\Psi(Q),$$

*where the sum ranges over all* **cd**-*monomials $w$ of degree $d$ that cover $S$ and contain exactly $|S|$* **d**'*s.*

We call the vector $(k_S)$, where $S$ ranges over all sparse subsets, the *flag $k$-vector.* As an example, let $d = 8$ and $S = \{0,3,5\}$. Then we have

$$k_{\{0,3,5\}} = [\mathbf{d}^3\mathbf{c}^2]\Psi(Q) + [\mathbf{d}^2\mathbf{cdc}]\Psi(Q) + [\mathbf{dcd}^2\mathbf{c}]\Psi(Q).$$

As a refinement of Proposition 1.3 in [16] we obtain the following relation.

**Proposition 6.6** *The coefficients of the* **cd**-*monomials containing $p$* **d**'*s can be expressed as an integer linear combination of $k_S$'s where $S$ has cardinality $p$. That is, for $w$ containing $p$* **d**'*s we have*

$$[w]\Psi(Q) = \sum_{|S|=p} q_{w,S} \cdot k_S,$$

*where the sum ranges over sparse sets $S$ and $q_{w,S}$ are integers.*

The proof follows by ordering the sets and the monomials by lexicographic order. It is then easy to see that the defining relation of $k_S$ corresponds to a lower triangular matrix with 1's on the main diagonal. Thus this linear relation is invertible over the integers.

**Lemma 6.7** *For $T$ a sparse subset of $\{0,1,\ldots,d-1\}$ we have that*

$$h_T = \sum_{U\subseteq T} k_U.$$

The proof is by expanding the **cd**-index in terms of **a**'s and **b**'s and collecting terms.

Combining Lemma 6.7 with the relation $f_S = \sum_{T\subseteq S} h_T$, we obtain

$$f_S = \sum_{U\subseteq S} 2^{|S\smallsetminus U|} \cdot k_U. \tag{6.1}$$

By the Principle of Inclusion-Exclusion the inverse of this relation is

$$k_S = \sum_{U\subseteq S} (-1)^{|S\smallsetminus U|} \cdot 2^{|S\smallsetminus U|} \cdot f_U. \tag{6.2}$$

**Lemma 6.8** *For a zonotope $Z$ we have that $2^{|S|}$ divides $f_S$.*

**Proof:** Observe that a zonotope is centrally symmetric and every face of a zonotope is a zonotope. Hence, every face of the zonotope $Z$ is centrally symmetric. (Zonotopes are characterized by the central symmetry of all their faces, in fact, of their 2-dimensional faces. See [5, Proposition 2.2.14].)

We may count $f_S$, where $S = \{i_1 < \cdots < i_k\}$, by first choosing a face $F_{i_k}$ of dimension $i_k$, then choosing an $i_{k-1}$-dimensional face of $F_{i_k}$, and so on. But since at each selection the face $F_{i_j}$ is centrally symmetric (including $Z$), we know that there is an even number of choices of $F_{i_{j-1}}$. By multiplying together all these factors of 2, we obtain $2^{|S|}$. ∎

**Lemma 6.9** *For a zonotope $Z$ we have that $k_S \equiv 0 \bmod 2^{|S|}$.*

**Proof:** It is enough to observe that $2^{|S|}$ divides $2^{|S \setminus U|} \cdot f_U$. ∎

By combining Proposition 6.6 and Lemma 6.9 we obtain

**Proposition 6.10** *The* **cd***-index of a zonotope $Z$ of dimension $d$ belongs to $\mathcal{T}_d$. That is, $\mathcal{S}_d \subseteq \mathcal{T}_d$.*

**Proof:** It is enough to prove for a zonotope $Z$ and a **cd**-monomial $w$ that the coefficient of $w$ in $\Psi(Z)$ is divisible by $2^{p(w)}$ where $p(w) = p$ is the number of **d**'s occurring in $w$. That is, $[w]\Psi(Z) \equiv 0 \bmod 2^p$.

Indeed, by Proposition 6.6 and Lemma 6.9 we have

$$[w]\Psi(Z) = \sum_{|S|=p} q_{w,S} \cdot k_S \equiv 0 \bmod 2^p,$$

where $S$ ranges over all sparse subsets of $\{0, 1, \ldots, d-1\}$ having cardinality $p$. ∎

Combining Propositions 6.4 and 6.10 gives us the main result of this section.

**Theorem 6.11** *The Abelian group generated by the* **cd***-indices of zonotopes of dimension $d$ is precisely $\mathcal{T}_d$, that is, all integral polynomials of degree $d$ in the variables* **c** *and* 2**d**.

As a direct consequence of this theorem, Proposition 6.6 and equation (6.2), we get the following.

**Corollary 6.12** *The lattice spanned by flag $f$-vectors of all $d$-zonotopes is the set of all integral vectors $f = \{f_S\}$ where $f_S$ is divisible by $2^{|S|}$.*

Since in the relation $f_S(Z) = f_{d-S}(\mathcal{H})$ between a $d$-zonotope $Z$ and its dual (essential) hyperplane arrangement $\mathcal{H}$ the sets $S$ and $d - S$ have the same cardinality, we obtain the following.

**Corollary 6.13** *The lattice spanned by flag $f$-vectors of all essential hyperplane arrangements in $\mathbb{R}^d$ is the set of all integral vectors $f = \{f_S\}$ where $f_S$ is divisible by $2^{|S|}$.*

# 7 Concluding remarks

Our method proves that zonotopes span, but is there a nice basis? We describe one possible basis, suggested in [14]. To do so, we define two operations $P$ and $B$ on a zonotope $Z$, where $PZ := \mathrm{Pri}(Z)$ and $BZ := M(\mathrm{Pri}(Z))$. Note that both result in a zonotope of one higher dimension. Now if we write a $BP$-word of length $d$, that is, a word of length $d$ made with the letters $B$ and $P$, we may view this as a sequence of $d$ operations performed on the 0-dimensional zonotope $\mathbf{0}$, and so as a $d$-dimensional zonotope. Liu [14] conjectured that a basis for the flag $f$-vectors (and hence, the **cd**-indices) of all $d$-dimensional zonotopes could be constructed by forming all $BP$-words of length $d$ ending in $P$ and having no two consecutive $B$'s. This should be compared to the basis for all polytopes given in [1] which was made up of similar combinations of pyramid and bipyramid operations.

We have described the lattice spanned by all **cd**-indices of zonotopes. The next natural problem is to determine all linear inequalities they must satisfy. It is known that the **cd**-index of any polytope must be nonnegative [16]. What more can be said about zonotopes? There is a family of linear inequalities known to be satisfied by flag $f$-vectors of zonotopes.

**Theorem 7.1 (Varchenko/Liu)** *If $Z$ is a $d$-dimensional zonotope and $S = \{i_1, \ldots, i_k\}$, then*

$$\frac{f_S(Z)}{f_{i_1}(Z)} < \binom{d - i_1}{i_2 - i_1, \ldots, i_k - i_{k-1}, d - i_k} \cdot 2^{i_k - i_1}.$$

For the case $k = 2$ this was proved in [19] (see also [5, §4.6]) and stated in [9]. For this generality a proof is given in [14]. Theorem 7.1 bounds the average number of $S \setminus \{i_1\}$ chains in links of $i_1$-faces of a $d$-dimensional zonotope by the number of $S \setminus \{i_1\}$ chains in a $(d - i_1)$-dimensional crosspolytope (all with the dimensions shifted appropriately). It is easy to find polytopes for which the inequalities in Theorem 7.1 fail. For example the cyclic polytope $C_d(n)$ does not satisfy the inequality for $S = \{0, 1\}$ when $n \geq 2d + 1$.

For 3-zonotopes it is enough to consider the pairs $(f_0, f_2)$. In this case the convex hull taken over all 3-zonotopes can be completely described as the cone with apex $(8, 6)$ (corresponding to the 3-cube), and extreme rays $(1, 1)$ and $(2, 1)$. Along the ray $(1, 1)$ can be found all zonotopes of the form $M^n$(Pri(square)), while on the other ray one finds all those of the form Pri($M^n$(square)) (prisms over even polygons). Other than the fact that only even points appear, the problem of determining which lattice points in this cone are actually realized by zonotopes (or by oriented matroids) appears to be a difficult one. See, for example, [10, Chap. 18].

The **cd**-index of a zonotope does depend only on the underlying matroid, and not on the oriented matroid. This suggests that there is a **cd**–index for matroids, in fact, a **c**-2**d**–index, independent of whether they are orientable or not. The authors are currently investigating the **cd**-index without reference to orientation.

For polytopes and certain classes of Eulerian posets the flag $h$-vector has been given a combinatorial interpretation. We wonder if the flag $k$-vector can be also given a combinatorial interpretation. Observe that we only define $k_S$ for sparse sets $S$. We may extend the flag $k$-vector to all sets by inverting the relation given in Lemma 6.7. It is known that the flag $h$-vector of a polytope, being the "fine" $h$-vector of a balanced Cohen-Macaulay complex, must be the fine $f$-vector of another balanced simplicial complex $\Delta$ [18, Thm 4.6]. Thus, in this case, the flag $k$-vector can be interpreted as the fine $h$-vector of this $\Delta$.

By the definition of the flag $k$-vector, $k_S \geq 0$ for all sparse $S$ whenever the **cd**-index is nonnegative. Is there a larger interesting class of posets for which the sparse flag $k$-vector is always nonnegative? For example, in the case just described this will occur if the complex $\Delta$ is itself Cohen-Macaulay; here the full flag $k$-vector will be nonnegative. That the full flag $k$-vector is not always nonnegative can be seen by examining the flag $k$-vector of a tetrahedron, for which $k_{\{0,1\}} = -4$.

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