1. Prove that the columns of an $m \times n$ matrix A are linearly independent if and only if the equation $A\vec{x} = \vec{0}$ has only the trivial solution. Note: this is an **if and only if** statement which means that you must prove **two** statements. You must prove:

   (a) If the columns of A are linearly independent, then $A\vec{x} = \vec{0}$ has only the trivial solution.

   (b) If $A\vec{x} = \vec{0}$ has only the trivial solution, then the columns of A are linearly independent.

   Proof.

2. An *affine transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ has the form $T(\vec{x}) = A\vec{x} + \vec{b}$, with A and $m \times n$ matrix and $\vec{b}$ in $\mathbb{R}^m$. Show that T is *not* a linear transformation when $\vec{b} \neq 0$. Proof.

3. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ be a linearly dependent set in $\mathbb{R}^n$. Prove that $\{T(\vec{v_1}), T(\vec{v_2}), T(\vec{v_3})\}$ is linearly dependent. Proof.

4. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that if T maps two linearly independent vectors onto a linearly dependent set, then $T(\vec{x}) = \vec{0}$ has a non-trivial solution. (*Hint:* Suppose $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ are linearly independent and yet $T(\vec{u})$ and $T(\vec{v})$ are linearly dependent. Then $c_1T(\vec{u}) + c_2T(\vec{v}) = \vec{0}$ for some weights c_1, c_2, not both zero. Use this equation.) Proof.

5. Suppose vectors $\vec{v_1}, \ldots, \vec{v_p}$ span $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Suppose $T(\vec{v_i}) = \vec{0}$ for $i = 0, \ldots, p$. Show that T is the zero transformation. That is, show that if $\vec{x}$ is any vector in $\mathbb{R}^n$, then $T(\vec{x}) = \vec{0}$. Proof.

1. Let $\vec{u}, \vec{v}$ be distinct (i.e., $\vec{u} \neq \vec{v}$) vectors in $\mathbb{R}^n$. Show that if $\{\vec{u}, \vec{v}\}$ forms a basis for a subspace V of $\mathbb{R}^n$ and a, b are non-zero scalars, then $\{\vec{u}+\vec{v}, a\vec{u}\}$ and $\{a\vec{u}, b\vec{v}\}$ are also bases for V. Proof.

2. Let S_1, S_2 be subsets of $\mathbb{R}^n$. Show that if $S_1 \subseteq S_2$ (this means that the subset S_1 is contained in the subset S_2, i.e., for each $\vec{x}$ in S_1, $\vec{x}$ is also in S_2), then $\text{Span}\{S_1\} \subseteq \text{Span}\{S_2\}$. In particular, if $S_1 \subseteq S_2$ and $\text{Span}\{S_1\} = \mathbb{R}^n$, then $\text{Span}\{S_2\} = \mathbb{R}^n$. Proof.

3. Let A be a matrix with eigenvalues λ_1, λ_2. Suppose $\lambda_1 \neq \lambda_2$ and the vectors $\vec{v_1}$ and $\vec{v_2}$ are eigenvectors corresponding to the eigenvalues λ_1, λ_2, respectively. Prove that the vectors $\vec{v_1}, \vec{v_2}$ are linearly independent. (Note: you must prove this from scratch, not by appealing to the more general theorem stated in class.) Proof.

4. Let $\vec{u}, \vec{v}$ be arbitrary vectors in $\mathbb{R}^3$. Show that

$$||\vec{u}+\vec{v}||^2 + ||\vec{u}-\vec{v}||^2 = 2||\vec{u}||^2 + 2||\vec{v}||^2.$$

(Note: in full generality, this statement holds for $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ for any integer n.) Proof.

5. Prove part of the *Cauchy-Schwarz inequality*: Let $\vec{u}, \vec{v}$ be two vectors in $\mathbb{R}^3$. Show that

$$|\vec{u} \cdot \vec{v}| = ||\vec{u}|| \, ||\vec{v}||$$

if and only if one vector is a scalar multiple of the other, i.e. $\vec{v} = c\vec{u}$ for some scalar c. Recall that an if and only if proof has two statements that you must prove:

   (a) If $\vec{v} = c\vec{u}$ for some scalar c, then $|\vec{u} \cdot \vec{v}| = ||\vec{u}|| \, ||\vec{v}||$.

   (b) If $|\vec{u} \cdot \vec{v}| = ||\vec{u}|| \, ||\vec{v}||$, then $\vec{v} = c\vec{u}$ for some scalar c.

(Note: more generally, this statement holds for $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ for any integer n.) Proof.