

# Functions

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# Flexible Concept of Function

Big Idea 1 from *Developing Essential Understandings of Functions Grades 9–12*:

The concept of function is intentionally broad and flexible, allowing it to apply to a wide range of situations. The notion of function encompasses many types of mathematical entities in addition to “classical” functions that describe quantities that vary continuously. For example, matrices and arithmetic and geometric sequences can be viewed as functions.

# Types of Functions

Find some examples of functions for each of the following types of functions.

# Types of Functions

1

**Input:** A real number

**Output:** A real number

# Types of Functions

2

**Input:** A real number

**Output:** An ordered pair of real numbers

# Types of Functions

3

**Input:** An ordered pair of real numbers

**Output:** A real number

# Types of Functions

4

**Input:** An ordered pair of real numbers

**Output:** An ordered pair of real numbers



# Types of Functions

5

**Input:** An ordered pair of real numbers

**Output:** A line

# Types of Functions

6

**Input:** An ordered triple of real numbers

**Output:** A parabola or a line

# Types of Functions

7

**Input:** A finite set of real numbers

**Output:** A real number

# Types of Functions

8

**Input:** A function

**Output:** A set of real numbers

# Types of Functions

9

**Input:** A polygon

**Output:** A real number

# Types of Functions

10

**Input:** A polynomial

**Output:** A polynomial

# Types of Functions

11

**Input:** A finite set of points in the plane

**Output:** A line

# Types of Functions

12

**Input:** A set of points in the plane

**Output:** A set of points in the plane



# Types of Functions

13

**Input:** A finite set

**Output:** An integer

# Types of Functions

14

**Input:** An ordered pair of real numbers

**Output:** An isometry

# Types of Functions

15

**Input:** A polygon in the plane and a linear function of two variables

**Output:** A number

# Types of Functions

16

**Input:** A line and a set of points in the plane

**Output:** A set of points in the plane

# Types of Functions

17

**Input:** A sequence of real numbers

**Output:** A real number

# Types of Functions

18

**Input:** A positive integer

**Output:** A real number integer

# Types of Functions

19

**Input:** A positive integer

**Output:** A set of positive integers

# Types of Functions

20

**Input:** A polyhedron

**Output:** An ordered triple of positive integers



# Types of Functions

21

**Input:** A polyhedron

**Output:** A real number

# Examples of Functions

For each of the following examples of functions, describe what type of function it is, using the preceding list.

# Examples of Functions

1

**Input:**  $x$

**Output:**  $9x - 3$

# Examples of Functions

2

**Input:**  $t$

**Output:**  $(3 \cos t, 3 \sin t)$

# Examples of Functions

3

**Input:**  $(x, y)$

**Output:**  $(x + \frac{4}{3}, y - 8)$

# Examples of Functions

4

**Input:**  $(x, y)$

**Output:**  $(3x - 2y, 5x + 9y)$

# Examples of Functions

5

**Input:**  $(x, y)$

**Output:**  $x^2 - y^2$

# Examples of Functions

6

**Input:**  $(a, b, c)$

**Output:**  $y = ax^2 + bx + c$



# Examples of Functions

7

**Input:**  $(x, y)$

**Output:**  $(y, x)$

# Examples of Functions

8

**Input:**  $x$

**Output:**  $2x^2 - \frac{3}{5}x + 5$

# Examples of Functions

9

**Input:**  $\{a_1, \dots, a_n\}$

**Output:** The median of  $a_1, \dots, a_n$

# Examples of Functions

10

**Input:**  $y = f(x)$

**Output:** The set of  $x$  for which  $f(x) = 0$

# Examples of Functions

11

**Input:**  $y = f(x)$

**Output:** The maximum value of  $f(x)$

# Examples of Functions

12

**Input:** A polygon

**Output:** Its perimeter

# Examples of Functions

13

**Input:** A polynomial

**Output:** Its derivative

# Examples of Functions

14

**Input:** A set of points in the plane

**Output:** A line of best fit



# Examples of Functions

15

**Input:**  $(x, y)$

**Output:**  $(5x, 5y)$

# Examples of Functions

16

**Input:**  $(x, y)$

**Output:**  $(-y, x)$

# Examples of Functions

17

**Input:** A set of points in the plane

**Output:** The rotation of this set by 90 degrees about the origin

# Examples of Functions

18

**Input:** A finite set

**Output:** The number of elements in the set

# Examples of Functions

19

**Input:**  $t$

**Output:**  $(4t - \frac{1}{2}, -2t + 3)$

# Examples of Functions

20

**Input:** A polynomial

**Output:** Its antiderivative (with constant term 0)

# Examples of Functions

21

**Input:**  $\{a_1, \dots, a_n\}$

**Output:** The mode of  $a_1, \dots, a_n$

# Examples of Functions

22

**Input:** A polyhedron

**Output:** Its volume



# Examples of Functions

23

**Input:**  $(m, b)$

**Output:** The reflection isometry over the line  $y = mx + b$

# Examples of Functions

24

**Input:** A polygon

**Output:** Its area

# Examples of Functions

25

**Input:**  $(a, b)$

**Output:** The translation isometry  $(x, y) \rightarrow (x + a, y + b)$

# Examples of Functions

26

**Input:** A polygon and a linear function  $ax + by$

**Output:** The maximum value of the function in the polygon

# Examples of Functions

27

**Input:** A line and a set of points in the plane

**Output:** The reflection of the the set in the line

# Examples of Functions

28

**Input:** A sequence of real numbers

**Output:** The limit of this sequence if it exists

# Examples of Functions

29

**Input:**  $(m, b)$

**Output:**  $y = mx + b$

# Examples of Functions

30

**Input:** A positive integer

**Output:** The set of its factors



# Examples of Functions

31

**Input:**  $\{a_1, \dots, a_n\}$

**Output:**  $\frac{a_1 + \dots + a_n}{n}$

# Examples of Functions

32

**Input:** A polyhedron

**Output:**  $(V, E, F)$

# Examples of Functions

33

**Input:** A set of points in the plane

**Output:** The reflection of this set in the  $y$ -axis

# Examples of Functions

34

**Input:** A finite set

**Output:** The number of subsets

# Examples of Functions

35

**Input:** A positive integer

**Output:** The number of its factors

# Examples of Functions

36

**Input:** A polyhedron

**Output:** Its surface area

# Examples of Functions

37

**Input:** A positive integer  $n$

**Output:** The sum  $1 + \cdots + n$

# Examples of Functions

38

**Input:**  $\theta$

**Output:** The point  $(r = \cos \theta, \theta)$  in polar coordinates



# Examples of Functions

39

**Input:**  $x$

**Output:**  $\sin x$

# Representing Functions

Big Idea 5 from *Developing Essential Understandings of Functions Grades 9–12*:

Functions can be represented in multiple ways, including algebraic (symbolic), graphical, verbal, and tabular representations. Links among these different representations are important to studying relationship and change.

# Representing Functions

An additional consideration: What technological tools can be used to effectively enhance the connections between multiple representations?

Try representing some of the previous examples using GeoGebra or WolframAlpha. (Include sequences of numbers, sequences of functions via differentiation, isometries, families of functions, linear programming, factoring positive integers, subsets, . . . .)

# Linear Functions

Discuss the questions in Reflect 1.42 from *Essential Understandings*. Then use GeoGebra to enact the various representations, and discuss whether or how this can enhance understanding.

# Linear Functions

Find representations for the function

$$y = 3x$$

where  $x$ , and  $y$  are real. What if  $x$  is restricted to be a nonnegative integer? What about other functions of the form  $y = ax$ , where  $a$  is real?

# Linear Functions

Find representations for the function

$$z = 2x + 3y$$

where  $x$ ,  $y$ , and  $z$  are real. What about other functions of the form  $z = ax + by$ , where  $a$  and  $b$  are real?

# Linear Functions

Find representations for the function

$$(x_2, y_2) = (3x_1, 3y_1)$$

where  $x_1$ ,  $x_2$ ,  $y_1$ , and  $y_2$  are real. Does this suggest a new representation for a function of the form  $y = 3x$ ? What about other functions of the form

$$(x_2, y_2) = (ax_1, ay_1)$$

where  $a$  is real?

# Linear Functions

Find representations for the function

$$z = 3w$$

where  $w$  and  $z$  are complex numbers. What about other functions of the form  $z = aw$  where  $a$  is real?



# Linear Functions

Find representations for the function

$$z = iw$$

where  $w$  and  $z$  are complex numbers. What about other functions of the form  $z = aw$  where  $a$  is complex?

# Sequences

Find representations for the function

$$f(n) = \sum_{i=1}^n i.$$

Consider both explicit and recursive representations. What about representing the Fibonacci sequence?

Try using WolframAlpha, <http://www.wolframalpha.com>, to guess the explicit formula.

# Isometries

Find representations for the following isometries:

- 1 Translation by the vector  $(2, -3)$ .
- 2 Reflections across the lines  $x = 0$ ,  $y = 0$ , or  $y = x$ .
- 3 (Counterclockwise) rotations by  $90^\circ$ ,  $180^\circ$ , or  $270^\circ$  about the origin.

# Isometries

What isometries are being represented by the following GeoGebra sketches:

- 1 transform3a.ggb
- 2 transform3b.ggb
- 3 transform3c.ggb
- 4 transform3d.ggb

# Matrices

Find representations for functions of the form

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} .$$

# Trigonometric Functions

## Motivations

- *Common Core State Standards*
- My 11th grade experience in public school
- Reasoning and making connections

# Trigonometric Functions

From the *Common Core State Standards*:

## Trigonometric Functions F-TF

Extend the domain of trigonometric functions using the unit circle

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.
3. (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for  $\pi/3$ ,  $\pi/4$  and  $\pi/6$ , and use the unit circle to express the values of sine, cosine, and tangent for  $\pi x$ ,  $\pi + x$ , and  $2\pi - x$  in terms of their values for  $x$ , where  $x$  is any real number.
4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

# Trigonometric Functions

## Trigonometric Functions F-TF

Prove and apply trigonometric identities

8. Prove the Pythagorean identity  $\sin^2(\theta) + \cos^2(\theta) = 1$  and use it to find  $\sin(\theta)$ ,  $\cos(\theta)$ , or  $\tan(\theta)$  given  $\sin(\theta)$ ,  $\cos(\theta)$ , or  $\tan(\theta)$  and the quadrant of the angle.
9. (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.



# Trigonometric Functions

## Similarity, Right Triangles, and Trigonometry G-SRT

Apply trigonometry to general triangles

9. (+) Derive the formula  $A = 1/2ab \sin(C)$  for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.
11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

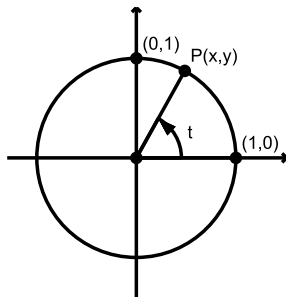
# Logical Connections

“Think deeply of simple things”

—Motto of the Ross Program at Ohio State

We want see connections, and we want things to make sense.

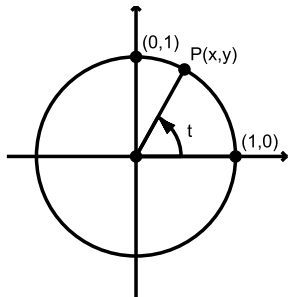
# The Unit Circle



Let's take as a *definition* that if  $P(x, y)$  is a point on the unit circle determined by angle  $t$ , then  $\cos t$  is defined to be  $x$  and  $\sin t$  is defined to be  $y$ . We can then define  $\tan t = \frac{\sin t}{\cos t}$ .

# Identities

Using no other knowledge of trigonometry than this, make sense of basic trig identities:



1  $\sin^2 t + \cos^2 t =$

2  $\sin(-t) =$

3  $\cos(-t) =$

4  $\sin(2\pi - t) =$

5  $\cos(2\pi - t) =$

6  $\sin(\pi - t) =$

7  $\cos(\pi - t) =$

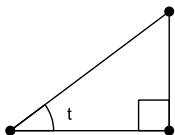
8  $\sin(\pi + t) =$

9  $\cos(\pi + t) =$

10  $\sin\left(\frac{\pi}{2} - t\right) =$

11  $\cos\left(\frac{\pi}{2} - t\right) =$

# Right Triangles

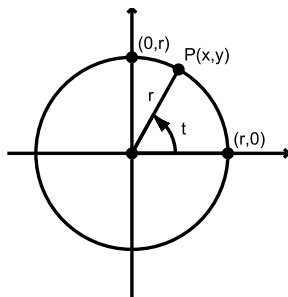


*Other* definitions for the trig functions of an acute angle  $t$  are:  
Consider any right triangle with acute angle  $t$ . Then

$$\sin t = \frac{\text{opposite side}}{\text{hypotenuse}}, \quad \cos t = \frac{\text{adjacent side}}{\text{hypotenuse}}, \quad \tan t = \frac{\text{opposite side}}{\text{adjacent side}}.$$

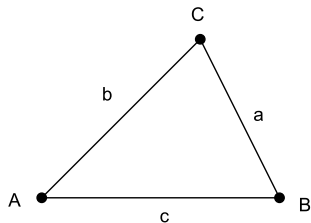
Explain why these definitions agree with the unit circle definitions when  $t$  is acute.

# Polar Coordinates



Explain why  $x = r \cos t$  and  $y = r \sin t$ .

# SAS Area of Triangle



Explain why the area of triangle  $\triangle ABC$  is given by

$$\text{area } \triangle ABC = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C.$$

Be sure your argument works even if one angle is right or obtuse.

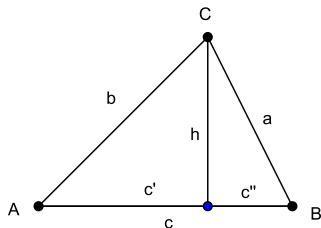
# Law of Sines

Prove that for any triangle  $\triangle ABC$ ,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$



# Law of Cosines

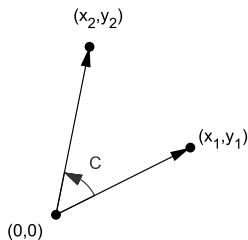


Derive the Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

Hint: Start with  $h^2 + (c'')^2 = a^2$ . Replace  $c''$  with  $c - c'$ , and replace  $h^2$  with  $b^2 - (c')^2$ . Be sure your argument works even if  $h$  falls outside  $\overline{AB}$ .

# A Cosine Formula

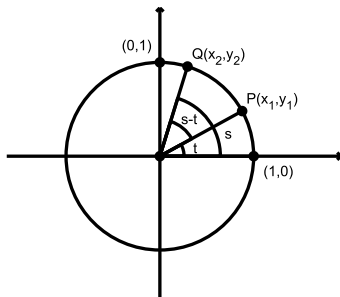


Use the Law of Cosines to prove that

$$\cos C = \frac{x_1 x_2 + y_1 y_2}{\sqrt{x_1^2 + y_1^2} \sqrt{x_2^2 + y_2^2}}.$$

Notice that neither orientation nor difference with  $2\pi$  make a difference.

# Angle Addition and Subtraction Formulas



Use the Cosine Formula to prove

$$\cos(s - t) = \cos s \cos t + \sin s \sin t.$$

Then prove

$$\cos(s + t) = \cos(s - (-t)) = \cos s \cos t - \sin s \sin t.$$

(Angles are oriented counterclockwise.)

# Angle Addition and Subtraction Formulas

Use  $\sin(s + t) = \cos\left(\frac{\pi}{2} - (s + t)\right) = \cos\left(\left(\frac{\pi}{2} - s\right) - t\right)$  to prove

$$\sin(s + t) = \sin s \cos t + \cos s \sin t.$$

Then prove

$$\sin(s - t) = \sin s \cos t - \cos s \sin t.$$

# Double Angle Formulas

Prove that

$$\sin 2t = 2 \sin t \cos t$$

and

$$\begin{aligned}\cos 2t &= \cos^2 t - \sin^2 t \\ &= 1 - 2 \sin^2 t \\ &= 2 \cos^2 t - 1.\end{aligned}$$

# Half Angle Formulas

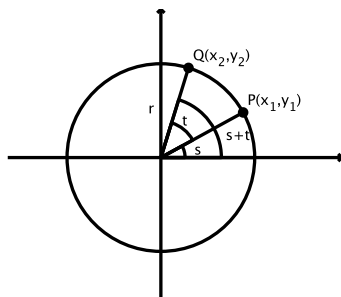
Prove that

$$\sin \frac{t}{2} = \sqrt{\frac{1 - \cos t}{2}}$$

and

$$\cos \frac{t}{2} = \sqrt{\frac{1 + \cos t}{2}}.$$

# Rotation Formula



(Angles are oriented counterclockwise.) Prove that

$$x_2 = x_1 \cos t - y_1 \sin t,$$

$$y_2 = x_1 \sin t + y_1 \cos t.$$

# Rotation Formula

In matrix form, this is

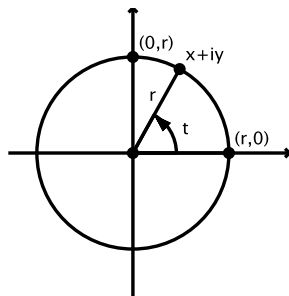
$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

Explain why the following formula holds (“undoing” the rotation):

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}.$$



# Complex Numbers



Explain why the complex number  $x + iy$  can be represented in the complex plane as the point  $(x, y)$ , and why it can also be expressed as  $r(\cos t + i \sin t)$  (sometimes abbreviated  $r \operatorname{cis} t$ ). (We call  $r$  the *modulus* of the complex number, and  $t$  its *argument*.)

# Complex Multiplication

Prove that multiplying two complex numbers  $r_1(\cos t_1 + i \sin t_1)$  and  $r_2(\cos t_2 + i \sin t_2)$  results in the complex number  $r_1 r_2(\cos(t_1 + t_2) + i \sin(t_1 + t_2))$ . That is to say, we multiply the moduli and add the angles (arguments).

# Complex Multiplication

- 1 Recognizing that the  $x$ -axis corresponds to the real numbers, show that  $i^2 = -1$ .
- 2 Geometrically and algebraically determine all complex numbers  $x$  such that  $x^4 = 1$ .
- 3 Geometrically and algebraically determine all complex numbers  $x$  such that  $x^6 = 1$ .
- 4 Geometrically and algebraically find all the square roots of  $i$ .

# Complex Multiplication

Motivation for the preceding from the *Common Core State Standards*:

The Complex Number System N-CN

Represent complex numbers and their operations on the complex plane.

4. (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.
5. (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. For example,  $(1 + \sqrt{3}i)^3 = 8$  because  $(1 + \sqrt{3}i)$  has modulus 2 and argument  $120^\circ$ .

# Transforming Functions and Functions for Transformations

Motivations;

- 1 *Essential Understandings*
- 2 *Common Core State Standards*
- 3 My 11th grade experience in public school

# Transforming

Big Idea 4 from *Developing Essential Understandings of Functions Grades 9–12*:

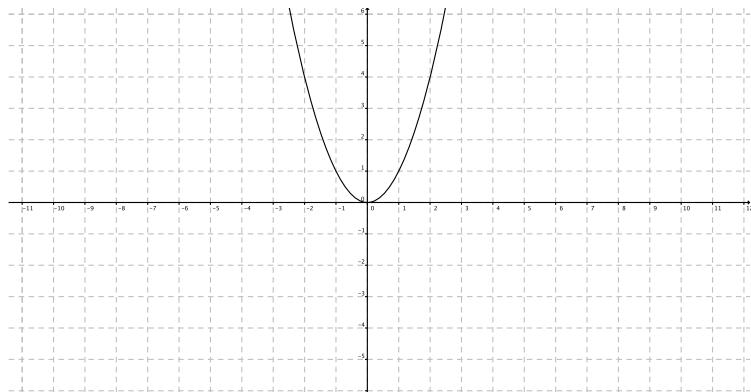
Functions can be combined by adding, subtracting, multiplying, dividing, and composing them. Functions sometimes have inverses. Functions can often be analyzed by viewing them as made from other functions.

# Transforming

## *Common Core State Standards:* High School — Geometry

The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation. Fundamental are the rigid motions: translations, rotations, reflections, and combinations of these, all of which are here assumed to preserve distance and angles (and therefore shapes generally). Reflections and rotations each explain a particular type of symmetry, and the symmetries of an object offer insight into its attributes—as when the reflective symmetry of an isosceles triangle assures that its base angles are congruent.

# Translation



Translate the given set by the vector  $(-2, 1)$ .



# Translation

If the set is defined by the equation  $y_1 = x_1^2$ , and the translation is defined by

$$x_2 = x_1 - 2,$$

$$y_2 = y_1 + 1,$$

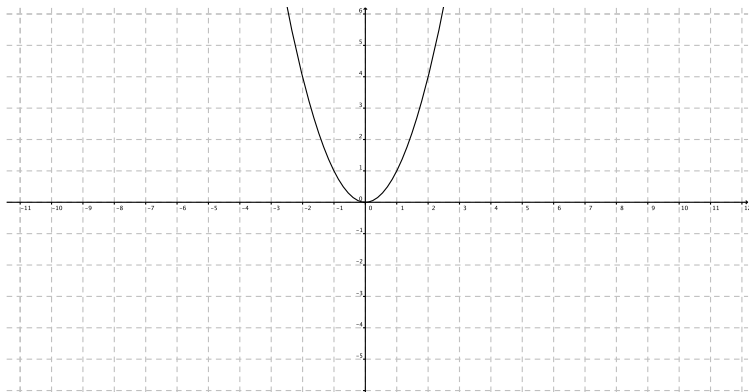
give the equation for the transformed set (in terms of  $x_2$  and  $y_2$ ).

# Translations

In general, if a set described by an equation in  $x_1$  and  $y_1$  is translated by the vector  $(h, k)$ , how do you get the equation of the transformed set?

Try this: What is the resulting equation when a unit circle about the origin is translated by the vector  $(3, -4)$ ?

# Dilations



Dilate the given set by a scaling factor of 2 using the origin as the center of dilation.

# Dilations

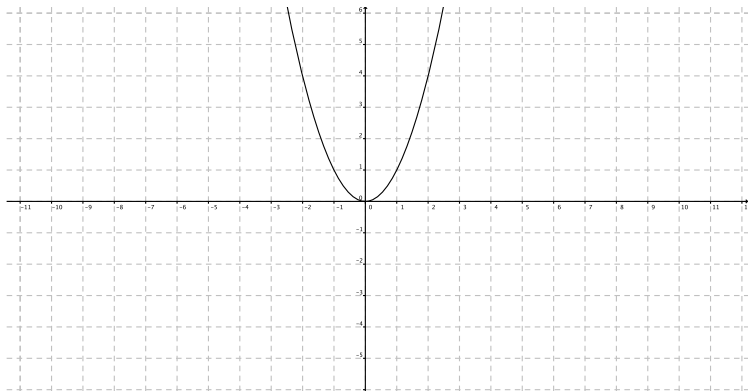
If the set is defined by the equation  $y_1 = x_1^2$ , and the dilation is defined by

$$x_2 = 2x_1,$$

$$y_2 = 2y_1,$$

give the equation for the transformed set.

# Scalings



Scale the given set with respect to the origin by a factor of 3 parallel to the  $x$ -axis and by a factor of 2 parallel to the  $y$ -axis.

# Scalings

If the set is defined by the equation  $y_1 = x_1^2$ , and the scaling is defined by

$$x_2 = 3x_1,$$

$$y_2 = 2y_1,$$

give the equation for the transformed set.

# Scalings

In general, if a set described by an equation in  $x_1$  and  $y_1$  is scaled with respect to the origin by a factor  $a$  parallel to the  $x$ -axis and by a factor of  $b$  parallel to the  $y$ -axis, how do you get the equation of the transformed set?

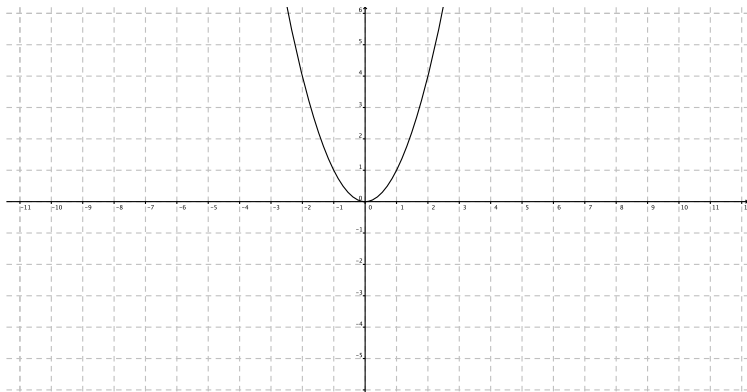
Try this: What is the resulting equation when a unit circle about the origin is scaled by a factor of  $\frac{1}{2}$  parallel to the  $x$ -axis and by a factor of 3 parallel to the  $y$ -axis?

# Translations and Scalings

Try this with GeoGebra: Make four sliders, say, for  $a$ ,  $b$ ,  $c$ , and  $d$ . Define a function  $f(x)$ , such as  $f(x) = \sin(x)$ . Then define  $g(x) = a * f(b * (x - c)) + d$ , and watch what happens when you move the sliders.



# Reflections



- 1 Reflect the given set across the  $x$ -axis.
- 2 Reflect the given set across the  $y$ -axis.
- 3 Reflect the given set across the line  $y = x$ .

# Reflections

Write formulas for each of the reflections.

If the set is defined by the equation  $y_1 = x_1^2$ , give the equations for the transformed sets.

# Reflections

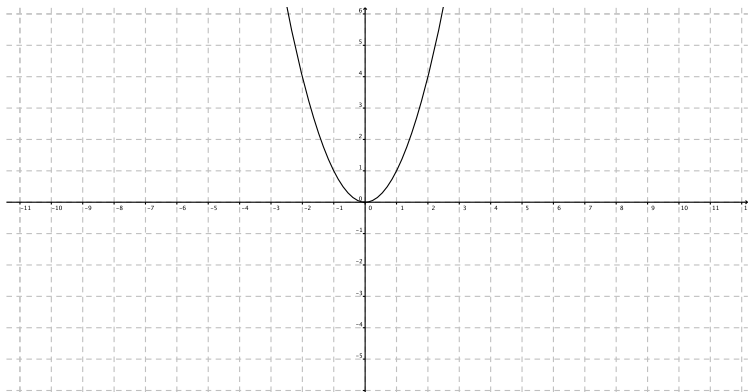
In general, if a set described by an equation in  $x_1$  and  $y_1$  is reflected across one of the previous lines, how do you get the equation of the transformed set?

# Reflections

What can you deduce from your answers about the symmetry of the set?

What can you deduce from your answers about finding the inverse of a function described by  $y = f(x)$ ?

# Rotations



- 1 Rotate the given set  $90^\circ$  counterclockwise about the origin.
- 2 Rotate the given set  $180^\circ$  counterclockwise about the origin.

# Rotations

Write formulas for each of the rotations.

If the set is defined by the equation  $y_1 = x_1^2$ , give the equations for the transformed sets.

# Rotations

In general, if a set described by an equation in  $x_1$  and  $y_1$  is rotated counterclockwise by one of the above angles, how do you get the equation of the transformed set?

# Rotations

Now try this: What is the resulting equation if the ellipse described by

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

is rotated counterclockwise  $45^\circ$  about the origin?



# Completing the Square

Consider the parabola given by the equation  $y_1 = 2x_1^2 - 12x_1 + 23$ . How can we translate it so that the vertex is at the origin? If the translation is given by  $x_2 = x_1 + h$  and  $y_2 = y_1 + k$ , then we have

$$y_2 - k = 2(x_2 - h)^2 - 12(x_2 - h) + 23$$

or

$$y_2 = 2x_2^2 + (-4h - 12)x_2 + 2h^2 + 12h + 23 + k.$$

We want  $-4h - 12$  to be zero, so  $h = -3$ .

We also want  $2h^2 + 12h + 23 + k = 0$  so  $k = -5$ .

Then  $y_2 = 2x_2^2$  is the equation of the translated parabola.

So the equation of the original parabola is  $y_1 - 5 = 2(x_1 - 3)^2$  which has vertex  $(3, 5)$ .

## More Rotations

Problem from my High School course:

What is the resulting equation if the parabola described by  $y_1 = x_1^2$  is rotated counterclockwise about the origin by the angle  $t$  having  $\sin t = \frac{7}{25}$  and  $\cos t = \frac{24}{25}$ ?

We use the rotation formulas derived earlier:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}.$$

Substituting, we have

$$-\frac{7}{25}x_2 + \frac{24}{25}y_2 = \left(\frac{24}{25}x_2 + \frac{7}{25}y_2\right)^2$$

which simplifies to

$$576x^2 + 336xy + 49y^2 + 175x - 600y = 0.$$

# Conics

Problem from my High School course:  
Analyze the conic given by the equation

$$73x^2 - 72xy + 52y^2 - 410x + 120y + 525 = 0.$$

We wish to apply a rotation by angle  $t$  that eliminates the  $xy$  term. We use the rotation formulas derived earlier (with  $x_1 = x$  and  $y_1 = y$ ). Let's abbreviate  $s = \sin t$  and  $c = \cos t$ .

$$\begin{aligned}x &= cx_2 + sy_2, \\y &= -sx_2 + cy_2.\end{aligned}$$

After substitution and simplification we find that the coefficient of  $x_2y_2$  is

$$42sc - 72(c^2 - s^2).$$

# Conics

We need an angle  $t$  so that this expression equals 0. Let  $T = 2t$ ,  $S = \sin T$ , and  $C = \cos T$ . Then  $S = 2sc$  and  $C = c^2 - s^2$  by the Double Angle Formulas. So we want an angle  $T$  with

$$21S - 72C = 0.$$

But this means  $\tan T = \frac{S}{C} = \frac{24}{7}$ . From this (and the Pythagorean Theorem) we calculate  $S = \frac{24}{25}$  and  $C = \frac{7}{25}$ .

# Conics

Now we use the Half Angle Formulas to find  $s$  and  $c$ :

$$s = \sqrt{\frac{1 - C}{2}} = \frac{3}{5},$$

$$c = \sqrt{\frac{1 + C}{2}} = \frac{4}{5}.$$

Using these values of  $c$  and  $s$ , the rotated conic has equation

$$100x_2^2 - 400x_2 + 25y_2^2 - 150y_2 + 525 = 0.$$

# Conics

Complete the two squares to get

$$100(x_2^2 - 4x_2 + 4) + 25(y_2^2 - 6y_2 + 9) = 100,$$

or

$$(x - 2)^2 + \frac{(y - 3)^2}{4} = 1.$$

This is an ellipse with center  $(2, 3)$ .

# Conics

So we have deduced that the original ellipse can be obtained from the ellipse  $x^2 + \frac{y^2}{4} = 1$  by first translating it by  $(2, 3)$  and then rotating it clockwise by the angle  $t$  with  $\sin t = \frac{3}{5}$  and  $\cos t = \frac{4}{5}$ .

# Modeling

## Motivations

- 1 *Common Core State Standards*
- 2 *Essential Understandings*



# Modeling

## *Common Core State Standards*

### Mathematical Practice 4: Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later.

# Modeling

They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

# Modeling

Big Idea 3 from *Developing Essential Understandings of Functions Grades 9–12*:

Functions can be classified into different families of functions, each with its own unique characteristics. Different families can be used to model different real-world phenomena.

# Fountains

Start GeoGebra and use the “Insert Image” command to paste a picture in the background of an arc of water, such as

[http://farm4.static.flickr.com/3258/2847369024\\_be4a45303a.jpg](http://farm4.static.flickr.com/3258/2847369024_be4a45303a.jpg).

Try entering, and then modifying (e.g., via the Algebra window, or by the various geometric transformations, such as translate and dilate—try using a slider), a function whose graph is a parabola, to match the shape of the arcs. You might try  $f(x) = a(b(x - c))^2 + d$  where  $a$ ,  $b$ ,  $c$ , and  $d$  are determined by sliders. Why is a parabola appropriate?

Bouncing balls also follow parabolic paths — find a stop-action photo to try, such as

<http://people.rit.edu/andpph/photofile-sci/bouncing-ball-7810.jpg>.

# St. Louis Arch

This time try to match an image of the St. Louis arch, which is a catenary, having an equation of the form  $y = \frac{a}{2}(e^{x/a} + e^{-x/a})$ .

Here is an image:

[http://3547.voxcdn.com/photos/0/34/46457\\_1.jpg](http://3547.voxcdn.com/photos/0/34/46457_1.jpg).

# Vibrating String

Try to match an image of a vibrating string with a sine function. Here is an image, but you have to rotate it 90 degrees with some other software before bringing it into GeoGebra:

<http://people.rit.edu/andpph/text-figures/strings/vibration-setup-8181a.jpg>.

# Nautilus

Try to match an image of the nautilus with a logarithmic spiral. Here is an image:

[http://www.gayot.com/blog/wp-content/uploads/2009/08/img\\_21831.jpg](http://www.gayot.com/blog/wp-content/uploads/2009/08/img_21831.jpg).

For the logarithmic spiral, given by  $r = ae^{b\theta}$ , make sliders for  $a$  and  $b$  and use the command

`Curve[a * e(bt) * cos(t), a * e(bt) * sin(t), t, 0, 25]`. Why does this create a logarithmic spiral?

# Sunflower

Now try to match a spiral to a sunflower. Here is an image:

[http://farm2.static.flickr.com/1278/694780262\\_8874b4f225.jpg](http://farm2.static.flickr.com/1278/694780262_8874b4f225.jpg).

How do you reverse the direction of the spiral?



# Curve Fitting

Go to the Data and Story Library (DASL),  
<http://lib.stat.cmu.edu/DASL/DataArchive.html>. Select  
“List All Topics”, then “Health”, then “Smoking and Cancer”. Select  
the Datafile. Highlight the six columns of the data set (omit the  
headers) and copy.

# Curve Fitting

Now open GeoGebra and select “Spreadsheet View” under “View”. Click on cell A1. Then paste the data in. Highlight the last three columns and press Delete to clear these cells. In cell D1 type “ $=\text{(B1,C1)}$ ”. This creates a point labeled D1, which will show up in the graphics view. Highlight cell D1. Then grab the lower right-hand corner and drag it down to create points D1 through D38. All of these points will show up in the graphics view. You may want to right-click the background of the graphics view to adjust the minimum and maximum values of the axes in order to see the points more clearly.

# Curve Fitting

To find the line of best fit, enter the following at the bottom of the graphics view: `FitLine[D1:D38]`. To find the correlation coefficient, enter `CorrelationCoefficient[D1:D38]`.

Now experiment with other data sets, and other “Fit” Commands, like `FitPoly[D1:D38,2]` or `FitPoly[D1:D38,3]`.

# Circular Motion

Let's use trig functions make a simplified model of a planet rotating on its axis while revolving around its sun.

Start in GeoGebra by drawing a unit circle centered at the origin, to represent the planet's orbit. Make a slider, say  $a$ , with min and max values 0 and 6.28, respectively. To place a point, say  $B$ , on this orbit (for the planet's center), enter  $(\cos(a), \sin(a))$ .

You can animate the slider by double-clicking on it, selecting "Animation On" under Basic, and "Increasing" under Slider. A small "play/pause" icon appears in the lower left of the view.

# Circular Motion

Now we will create a planet by pausing the animation and drawing a circle with center  $B$  and radius 0.1. You can then select the circle and fill its interior with a color if you wish.

Place a reference point, say,  $C$ , on the planet by entering  $B + .1 * (\cos(a), \sin(a))$ .

Turn the animation back on. You may wish to pause and adjust the speed and the increment of the slider for smoother motion.

# Circular Motion

Notice that right now, the planet rotates exactly once about its axis while it revolves once about the sun. The net effect for the inhabitants of the planet is that the sun does not move across the sky! (This is what our moon is doing with respect to the Earth.)

We can change this by changing the definition of  $C$  to be  $B + .1 * (\cos(3a), \sin(3a))$ . Now the planet rotates around its axis three times during the year. Yet the inhabitants experience only two days — why is that? Experiment with other numbers of days in the year.

Can you add a moon?

# Some Technology Resources

- Mathematics Education and Outreach Resources, <http://www.ms.uky.edu/~lee/outreach/outreach.html>. My master list of useful resources.
- GeoGebra, <http://www.geogebra.org>, for geometric and algebraic investigations.
- GeoGebra 5.0, [http://www.geogebra.org/en/wiki/index.php/Release\\_Notes\\_GeoGebra\\_5.0](http://www.geogebra.org/en/wiki/index.php/Release_Notes_GeoGebra_5.0), a beta version of GeoGebra including 3D features.
- Google SketchUp, <http://sketchup.google.com>, for three-dimensional modeling.
- WolframAlpha, <http://www.wolframalpha.com>, for online calculation, including graphing, symbolic algebra.
- Wolfram Demonstrations Project, <http://demonstrations.wolfram.com>, for many illustrative applets.

# Some Technology Resources

- National Library of Virtual Manipulatives, <http://nlvm.usu.edu>, for K–12 applets.
- NCTM's Illuminations, <http://illuminations.nctm.org>, for activities and lessons.
- Interactivate, <http://www.shodor.org/interactivate>, for activities and lessons.
- Data and Story Library, <http://lib.stat.cmu.edu/DASL>, for many interesting data sets.
- Common Core State Standards for Mathematics, <http://www.corestandards.org/the-standards>.
- Jing, <http://www.techsmith.com/Jing>, for capturing screen images and making videos.
- Dropbox, <http://www.dropbox.com>, for storing and sharing files with others.