Proof of Euler's Formula for Convex Polyhedra

Imagine that the polyhedron is in a tub that is being filled with water. Hold the polyhedron in such a way that no two vertices are at the same horizontal level. We will count each vertex at the moment when the water level reaches it. We will count each edge at the moment when the water level reaches its higher endpoint. We will count each face at the moment when the water level reaches its top-most vertex.

When the water level reaches the bottom-most vertex of the polyhedron, we count only this vertex, but no edge or face has been submerged, so no edge or face is counted yet. So at this moment, $V = 1$, $E = 0$, and $F = 0$, and thus $V - E + F = 1$.

Consider when the water level reaches any other vertex of the polyhedron except the top-most one. This vertex gets counted at this moment. Suppose there are exactly k edges from this vertex that extend downward into the water. Then these k edges get counted at this moment. Also, in between these k edges are $k - 1$ faces of the polyhedron that have just become submerged, so we count these $k - 1$ faces at this moment. So altogether V increases by 1, E increases by k, and F increases by $k - 1$. Thus $V - E + F$ changes by $1 - k + (k - 1) = 0$; i.e., $V - E + F$ does not change and remains equal to 1.

Finally, consider when the water level reaches the top-most vertex of the polyhedron. This vertex gets counted at this moment. All of the edges from this vertex extend downward into the water and are counted at this moment. Let's say that there are k of them. Also, in between these k edges there are k faces of the polyhedron, and all these get counted at this moment. So altogether V increases by 1, E increases by k, and F increases by k. Thus $V - E + F$ changes by $1 - k + k = 1$; i.e., $V - E + F$ increases from its old value of 1 to its final value of 2.