A&S 153
#6 Euler's Relation

Part 1: Initial Investigations

1. A tetrahedron has 6 edges. Try to construct a polyhedron with exactly 7 edges.

2. Try to construct a polyhedron for which every face has at least 6 sides.

Part 2: Some Basic Inequalities

1. Euler's Formula says that for every convex polyhedron,

$$V - E + F = 2. \tag{1}$$

2. Let F_i denote the number of faces that have i vertices (and hence i edges). Explain why

$$3F_3 + 4F_4 + 5F_5 + 6F_6 + \cdots = 2E. \tag{2}$$

3. Explain why

$$3F_3 + 4F_4 + 5F_5 + 6F_6 + \cdots \geq 3F_3 + 3F_4 + 3F_5 + 3F_6 + \cdots = 3F. \tag{3}$$

4. Conclude

$$2E \geq 3F. \tag{4}$$

5. Let V_i denote the number of vertices at which i faces (and hence i edges) meet. Prove

$$3V_3 + 4V_4 + 5V_5 + 6V_6 + \cdots = 2E. \tag{5}$$

6. Prove

$$2E \geq 3V. \tag{6}$$

7. Use (1) and (4) to prove

$$F \leq 2V - 4. \tag{7}$$

8. Use (1) and (6) to prove

$$F \geq \frac{1}{2}V + 2. \tag{8}$$

Part 3: Enumerating Possibilities

1. Label the horizontal axis in a coordinate system V and the vertical axis F. Graph the region for which the above two inequalities (7) and (8) hold. Begin listing the whole number solutions in the table below

V	F
4	4
5	5
5	6

Can you find a formula for the number of different possible values of F for a given value of V?

2. Prove that no polyhedron has exactly 7 edges.

3. Think of ways to construct polyhedra that match all possible values of V and F in the above table. For example, a tetrahedron has $(V, F) = (4, 4)$ and a pyramid with a pentagonal base has $(V, F) = (5, 5)$. If you chop off one corner of a tetrahedron, the resulting polyhedron has $(V, F) = (6, 5)$. If you build a shallow pyramid over one of the triangles of a tetrahedron, the resulting polyhedron has $(V, F) = (5, 6)$.

Part 4: Some More Inequalities

1. Use (1) and (4) to prove that
$$6 \leq 3V - E. \tag{9}$$

2. Use (1) and (6) to prove that
$$6 \leq 3F - E. \tag{10}$$

3. Use (10) and one of the earlier formulas to prove that
$$12 \leq 3F_3 + 2F_4 + 1F_5 + 0F_6 - 1F_7 - 2F_8 - \cdots. \tag{11}$$

4. From this, prove that every polyhedron must have at least one face that is a triangle, quadrilateral, or pentagon.

5. Prove that every polyhedron must have at least one vertex at which exactly 3, 4, or 5 edges meet.

6. A truncated icosahedron (soccer ball) is an example of a polyhedron such that (1) each face is a pentagon or a hexagons, and (2) exactly three faces meet at each vertex. Prove that any polyhedron with these two properties must have exactly 12 pentagons. Can you think of a polyhedron that has 12 pentagons but a different number of hexagons than a truncated icosahedron (which has 20 hexagons)?

Part 5: Angle Deficit

You may remember from plane geometry that for any polygon, the sum of the exterior angles (the amount by which the interior angle falls short of 180 degrees) always equals 360 degrees. There is a similar formula for polyhedra.

For each vertex we will calculate by how much the sum of the interior angles of the polygons meeting there falls short of 360 degrees. Then we will sum these shortfalls over all the vertices.

1. Explain why the sum of all the shortfalls will equal $360V - S$ where S is the sum of all of the interior angles of all the polygons.

2. Remember that the sum of the interior angles of a polygon with n sides is $(n-2)180$ degrees. Prove that

$$S = 180(F_3 + 2F_4 + 3F_5 + 4F_6 + \cdots). \tag{12}$$

3. Now use (1) and (2) to show that

$$360V - S = 720 \tag{13}$$

for all polyhedra.