Functions

Carl Lee
University of Kentucky
http://www.ms.uky.edu/~lee

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http://www.ms.uky.edu/~lee/ckhs2011/ckhs2011.html

Flexible Concept of Function

Big Idea 1 from *Developing Essential Understandings of Functions Grades 9–12*:

The concept of function is intentionally broad and flexible, allowing it to apply to a wide range of situations. The notion of function encompasses many types of mathematical entities in addition to "classical" functions that describe quantities that vary continuously. For example, matrices and arithmetic and geometric sequences can be viewed as functions.

Types of Functions

Find some examples of functions for each of the following types of functions.

Types of Functions

1

Input: A real number

Output: A real number

Types of Functions

2

Input: A real number

Output: An ordered pair of real numbers

Types of Functions

3

Input: An ordered pair of real numbers

Output: A real number

Types of Functions

4

Input: An ordered pair of real numbers

Output: An ordered pair of real numbers

Types of Functions

5

Input: An ordered pair of real numbers

Output: A line

Types of Functions

6

Input: An ordered triple of real numbers

Output: A parabola or a line

Types of Functions

7

Input: A finite set of real numbers

Output: A real number

Types of Functions

8

Input: A function

Output: A set of real numbers

Types of Functions

9

Input: A polygon

Output: A real number

Types of Functions

10

Input: A polynomial

Output: A polynomial

Types of Functions

11

Input: A finite set of points in the plane

Output: A line

Types of Functions

12
Input: A set of points in the plane

Output: A set of points in the plane

Types of Functions

13

Input: A finite set

Output: An integer

Types of Functions

14

Input: An ordered pair of real numbers

Output: An isometry

Types of Functions

15

Input: A polygon in the plane and a linear function of two variables

Output: A number

Types of Functions

16

Input: A line and a set of points in the plane

Output: A set of points in the plane

Types of Functions

17

Input: A sequence of real numbers

Output: A real number

Types of Functions

18

Input: A positive integer

Output: A real number integer

Types of Functions

19

Input: A positive integer

Output: A set of positive integers

Types of Functions

20
Input: A polyhedron

Output: An ordered triple of positive integers

Types of Functions

21

Input: A polyhedron

Output: A real number

Examples of Functions

For each of the following examples of functions, describe what type of function it is, using the preceding list.

Examples of Functions

1

Input: x

Output: $9x - 3$

Examples of Functions

2
Input: t

Output: $(3\cos t, 3\sin t)$

Examples of Functions

3

Input: (x, y)

Output: $(x + \frac{4}{3}, y - 8)$

Examples of Functions

4

Input: (x, y)

Output: $(3x - 2y, 5x + 9y)$

Examples of Functions

5

Input: (x, y)

Output: $x^2 - y^2$

Examples of Functions

6

Input: (a, b, c)

Output: $y = ax^2 + bx + c$

Examples of Functions

7

Input: (x, y)

Output: (y, x)

Examples of Functions

8

Input: x

Output: $2x^2 - \frac{3}{5}x + 5$

Examples of Functions

9
Input: $\{a_1, \ldots, a_n\}$

Output: The median of $a_1, \ldots, a_n$

Examples of Functions

10

Input: $y = f(x)$

Output: The set of x for which $f(x) = 0$

Examples of Functions

11

Input: $y = f(x)$

Output: The maximum value of $f(x)$

Examples of Functions

12

Input: A polygon

Output: Its perimeter

Examples of Functions

13

Input: A polynomial

Output: Its derivative

Examples of Functions

14

Input: A set of points in the plane

Output: A line of best fit

Examples of Functions

15

Input: (x, y)

Output: $(5x, 5y)$

Examples of Functions

16

Input: (x, y)

Output: $(-y, x)$

Examples of Functions

17
Input: A set of points in the plane

Output: The rotation of this set by 90 degrees about the origin

Examples of Functions

18

Input: A finite set

Output: The number of elements in the set

Examples of Functions

19

Input: t

Output: $(4t - \frac{1}{2}, -2t + 3)$

Examples of Functions

20

Input: A polynomial

Output: Its antiderivative (with constant term 0)

Examples of Functions

21

Input: $\{a_1, \ldots, a_n\}$

Output: The mode of $a_1, \ldots, a_n$

Examples of Functions

22
Input: A polyhedron

Output: Its volume

Examples of Functions

23

Input: (m, b)

Output: The reflection isometry over the line $y = mx + b$

Examples of Functions

24

Input: A polygon

Output: Its area

Examples of Functions

25

Input: (a, b)

Output: The translation isometry $(x, y) \to (x + a, y + b)$

Examples of Functions

26

Input: A polygon and a linear function $ax + by$

Output: The maximum value of the function in the polygon

Examples of Functions

27

Input: A line and a set of points in the plane

Output: The reflection of the the set in the line

Examples of Functions

28

Input: A sequence of real numbers

Output: The limit of this sequence if it exists

Examples of Functions

29

Input: (m, b)

Output: $y = mx + b$

Examples of Functions

30

Input: A positive integer

Output: The set of its factors

Examples of Functions

31

Input: $\{a_1, \ldots, a_n\}$

Output: $\frac{a_1+\cdots+a_n}{n}$

Examples of Functions

32
Input: A polyhedron

Output: (V, E, F)

Examples of Functions

33

Input: A set of points in the plane

Output: The reflection of this set in the y-axis

Examples of Functions

34

Input: A finite set

Output: The number of subsets

Examples of Functions

35

Input: A positive integer

Output: The number of its factors

Examples of Functions

36

Input: A polyhedron

Output: Its surface area

Examples of Functions

37

Input: A positive integer n

Output: The sum $1 + \cdots + n$

Examples of Functions

38

Input: θ

Output: The point $(r = \cos\theta, \theta)$ in polar coordinates

Examples of Functions

39

Input: x

Output: $\sin x$

Representing Functions

Big Idea 5 from *Developing Essential Understandings of Functions Grades 9–12*:

Functions can be represented in multiple ways, including algebraic (symbolic), graphical, verbal, and tabular representations. Links among these different representations are important to studying relationship and change.

Representing Functions

An additional consideration: What technological tools can be used to effective enhance the connections between multiple representations?

Try representing some of the previous examples using GeoGebra or WolframAlpha. (Include sequences of numbers, sequences of functions via differentiation, isometries, families of functions, linear programming, factoring positive integers, subsets, ...)

Linear Functions

Discuss the questions in Reflect 1.42 from *Essential Understandings.* Then use GeoGebra to enact the various representations, and discuss whether or how this can enhance understanding.

Linear Functions

Find representations for the function

$$y = 3x$$

where x, and y are real. What if x is restricted to be a nonnegative integer? What about other functions of the form $y = ax$, where a is real?

Linear Functions

Find representations for the function

$$z = 2x + 3y$$

where x, y, and z are real. What about other functions of the form $z = ax + by$, where a and b are real?

Linear Functions

Find representations for the function

$$(x_2, y_2) = (3x_1, 3y_1)$$

where x_1, x_2, y_1, and y_2 are real. Does this suggest a new representation for a function of the form $y = 3x$? What about other functions of the form

$$(x_2, y_2) = (ax_1, ay_1)$$

where a is real?

Linear Functions

Find representations for the function

$$z = 3w$$

where w and z are complex numbers. What about other functions of the form $z = aw$ where a is real?

Linear Functions

Find representations for the function

$$z = iw$$

where w and z are complex numbers. What about other functions of the form $z = aw$ where a is complex?

Sequences

Find representations for the function

$$f(n) = \sum_{i=1}^{n} i.$$

Consider both explicit and recursive representations. What about representing the Fibonacci sequence?

Try using WolframAlpha, `http://www.wolframalpha.com`, to guess the explicit formula.

Isometries

Find representations for the following isometries:

1. Translation by the vector $(2, -3)$.
2. Reflections across the lines $x = 0$, $y = 0$, or $y = x$.
3. (Counterclockwise) rotations by 90°, 180°, or 270° about the origin.

Isometries

What isometries are being represented by the following GeoGebra sketches:

1. transform3a.ggb
2. transform3b.ggb
3. transform3c.ggb
4. transform3d.ggb

Matrices

Find representations for functions of the form

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

Trigonometric Functions

Motivations

- *Common Core State Standards*
- My 11th grade experience in public school
- Reasoning and making connections

Trigonometric Functions

From the *Common Core State Standards:*

Trigonometric Functions F-TF

Extend the domain of trigonometric functions using the unit circle

1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.
3. (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosine, and tangent for πx, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.
4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

Trigonometric Functions

Trigonometric Functions F-TF

Prove and apply trigonometric identities

8. Prove the Pythagorean identity $\sin^2(\theta) + cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.
9. $(+)$ Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

Trigonometric Functions

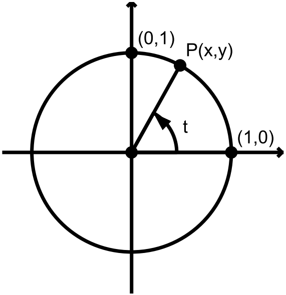
Similarity, Right Triangles, and Trigonometry G-SRT
Apply trigonometry to general triangles

9. (+) Derive the formula $A = 1/2ab\sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
10. (+) Prove the Laws of Sines and Cosines and use them to solve problems.
11. (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Logical Connections

"Think deeply of simple things"
—Motto of the Ross Program at Ohio State
We want see connections, and we want things to make sense.

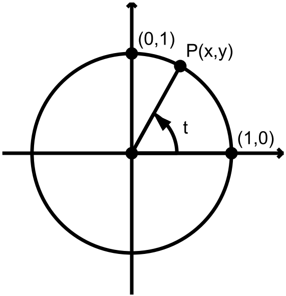
The Unit Circle



Let's take as a *definition* that if $P(x, y)$ is a point on the unit circle determined by angle t, then $\cos t$ is defined to be x and $\sin t$ is defined to be y. We can then define $\tan t = \frac{\sin t}{\cos t}$.

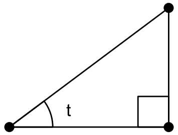
Identities

Using no other knowledge of trigonometry than this, make sense of basic trig identities:



1. $\sin^2 t + \cos^2 t =$
2. $\sin(-t) =$
3. $\cos(-t) =$
4. $\sin(2\pi - t) =$
5. $\cos(2\pi - t) =$
6. $\sin(\pi - t) =$
7. $\cos(\pi - t) =$
8. $\sin(\pi + t) =$
9. $\cos(\pi + t) =$
10. $\sin(\frac{\pi}{2} - t) =$
11. $\cos(\frac{\pi}{2} - t) =$

Right Triangles

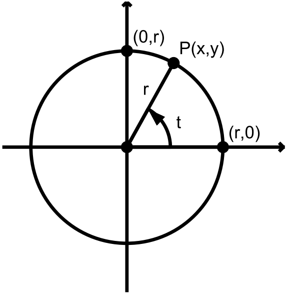


Other definitions for the trig functions of an acute angle t are: Consider any right triangle with acute angle t. Then

$$\sin t = \frac{\text{opposite side}}{\text{hypotenuse}}, \quad \cos t = \frac{\text{adjacent side}}{\text{hypotenuse}}, \quad \tan t = \frac{\text{opposite side}}{\text{adjacent side}}.$$

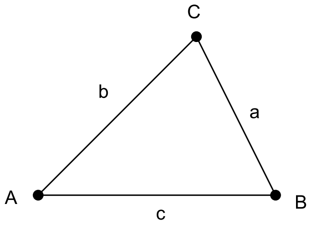
Explain why these definitions agree with the unit circle definitions when t is acute.

Polar Coordinates



Explain why $x = r\cos t$ and $y = r\sin t$.

SAS Area of Triangle



Explain why the area of triangle ΔABC is given by

$$\text{area } \Delta ABC = \frac{1}{2} bc \sin A = \frac{1}{2} ac \sin B = \frac{1}{2} ab \sin C.$$

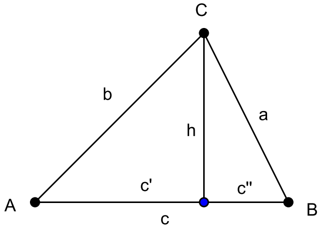
Be sure your argument works even if one angle is right or obtuse.

Law of Sines

Prove that for any triangle ΔABC,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Law of Cosines

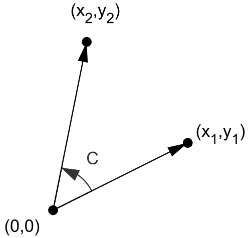


Derive the Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Hint: Start with $h^2 + (c'')^2 = a^2$. Replace c'' with $c - c'$, and replace h^2 with $b^2 - (c')^2$. Be sure your argument works even if h falls outside $\overline{AB}$.

A Cosine Formula

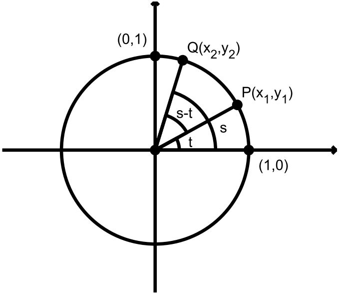


Use the Law of Cosines to prove that

$$\cos C = \frac{x_1 x_2 + y_1 y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}.$$

Notice that neither orientation nor difference with 2π make a difference.

Angle Addition and Subtraction Formulas



Use the Cosine Formula to prove

$$\cos(s - t) = \cos s \cos t + \sin s \sin t.$$

Then prove

$$\cos(s + t) = \cos(s - (-t)) = \cos s \cos t - \sin s \sin t.$$

(Angles are oriented counterclockwise.)

Angle Addition and Subtraction Formulas

Use $\sin(s+t) = \cos(\frac{\pi}{2} - (s+t)) = \cos((\frac{\pi}{2} - s) - t)$ to prove

$$\sin(s+t) = \sin s \cos t + \cos s \sin t.$$

Then prove

$$\sin(s-t) = \sin s \cos t - \cos s \sin t.$$

Double Angle Formulas

Prove that

$$\sin 2t = 2 \sin t \cos t$$

and

$$\begin{aligned} \cos 2t &= \cos^2 t - \sin^2 t \\ &= 1 - 2\sin^2 t \\ &= 2\cos^2 t - 1. \end{aligned}$$

Half Angle Formulas

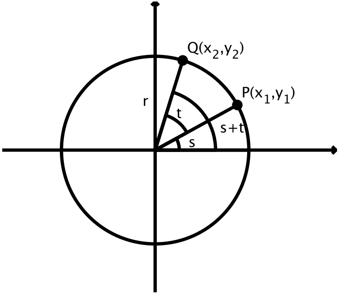
Prove that

$$\sin\frac{t}{2} = \sqrt{\frac{1-\cos t}{2}}$$

and

$$\cos\frac{t}{2} = \sqrt{\frac{1+\cos t}{2}}.$$

Rotation Formula



(Angles are oriented counterclockwise.) Prove that

$$\begin{aligned} x_2 &= x_1 \cos t - y_1 \sin t, \\ y_2 &= x_1 \sin t + y_1 \cos t. \end{aligned}$$

Rotation Formula

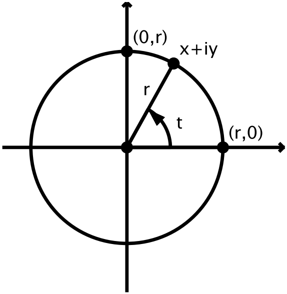
In matrix form, this is

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

Explain why the following formula holds ("undoing" the rotation):

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}.$$

Complex Numbers



Explain why the complex number $x + iy$ can be represented in the complex plane as the point (x, y), and why it can also be expressed as $r(\cos t + i \sin t)$ (sometimes abbreviated $r \operatorname{cis} t$). (We call r the *modulus* of the complex number, and t its *argument*.)

Complex Multiplication

Prove that multiplying two complex numbers $r_1(\cos t_1 + i \sin t_1)$ and $r_2(\cos t_2 + i \sin t_2)$ results in the complex number $r_1 r_2(\cos(t_1 + t_2) + i \sin(t_1 + t_2))$. That is to say, we multiply the moduli and add the angles (arguments).

Complex Multiplication

1. Recognizing that the x-axis corresponds to the real numbers, show that $i^2 = -1$.
2. Geometrically and algebraically determine all complex numbers x such that $x^4 = 1$.
3. Geometrically and algebraically determine all complex numbers x such that $x^6 = 1$.
4. Geometrically and algebraically find all the square roots of i.

Complex Multiplication

Motivation for the preceding from the *Common Core State Standards*:

The Complex Number System N-CN

Represent complex numbers and their operations on the complex plane.

4. (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.
5. (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. For example, $(1+\sqrt{3}i)^3 = 8$ because $(1+\sqrt{3}i)$ has modulus 2 and argument 120°.

Transforming Functions and Functions for Transformations

Motivations;

1. *Essential Understandings*
2. *Common Core State Standards*
3. My 11th grade experience in public school

Transforming

Big Idea 4 from *Developing Essential Understandings of Functions Grades 9–12*:

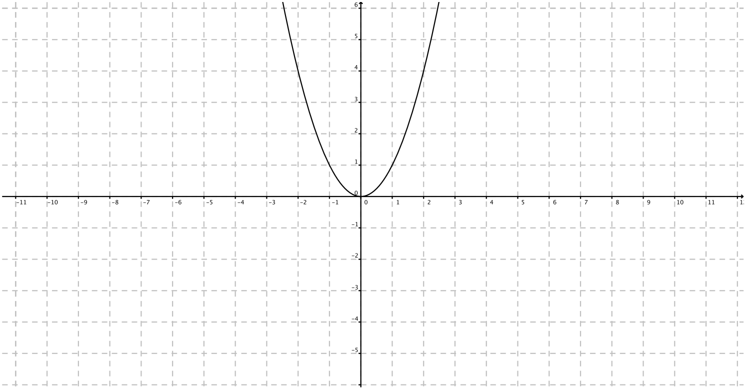
Functions can be combined by adding, subtracting, multiplying, dividing, and composing them. Functions sometimes have inverses. Functions can often be analyzed by viewing them as made from other functions.

Transforming

Common Core State Standards:
High School — Geometry

The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation. Fundamental are the rigid motions: translations, rotations, reflections, and combinations of these, all of which are here assumed to preserve distance and angles (and therefore shapes generally). Reflections and rotations each explain a particular type of symmetry, and the symmetries of an object offer insight into its attributes–as when the reflective symmetry of an isosceles triangle assures that its base angles are congruent.

Translation



Translate the given set by the vector $(-2, 1)$.

Translation

If the set is defined by the equation $y_1 = x_1^2$, and the translation is defined by

$$x_2 = x_1 - 2,$$
$$y_2 = y_1 + 1,$$

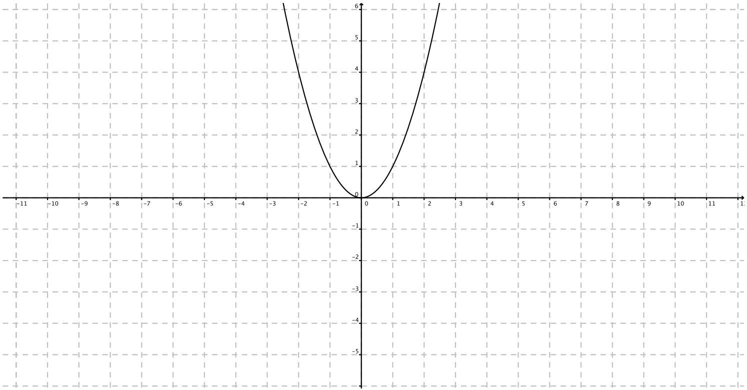
give the equation for the transformed set (in terms of x_2 and y_2).

Translations

In general, if a set described by an equation in x_1 and y_1 is translated by the vector (h, k), how do you get the equation of the transformed set?

Try this: What is the resulting equation when a unit circle about the origin is translated by the vector $(3, -4)$?

Dilations



Dilate the given set by a scaling factor of 2 using the origin as the center of dilation.

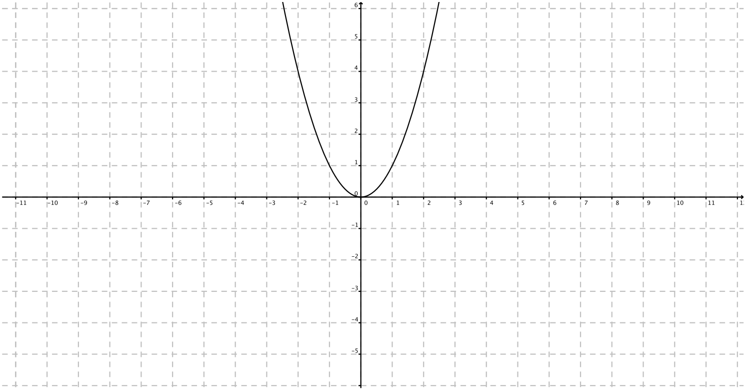
Dilations

If the set is defined by the equation $y_1 = x_1^2$, and the dilation is defined by

$$x_2 = 2x_1,$$
$$y_2 = 2y_1,$$

give the equation for the transformed set.

Scalings



Scale the given set with respect to the origin by a factor of 3 parallel to the x-axis and by a factor of 2 parallel to the y-axis.

Scalings

If the set is defined by the equation $y_1 = x_1^2$, and the scaling is defined by

$$\begin{aligned} x_2 &= 3x_1, \\ y_2 &= 2y_1, \end{aligned}$$

give the equation for the transformed set.

Scalings

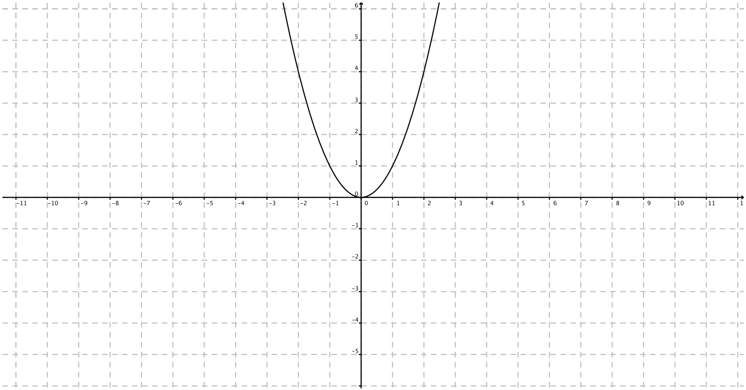
In general, if a set described by an equation in x_1 and y_1 is scaled with respect to the origin by a factor a parallel to the x-axis and by a factor of b parallel to the y-axis, how do you get the equation of the transformed set?

Try this: What is the resulting equation when a unit circle about the origin is scaled by a factor of $\frac{1}{2}$ parallel to the x-axis and by a factor of 3 parallel to the y-axis?

Translations and Scalings

Try this with GeoGebra: Make four sliders, say, for a, b, c, and d. Define a function $f(x)$, such as $f(x) = \sin(x)$. Then define $g(x) = a * f(b * (x - c)) + d$, and watch what happens when you move the sliders.

Reflections



1. Reflect the given set across the x-axis.
2. Reflect the given set across the y-axis.
3. Reflect the given set across the line $y = x$.

Reflections

Write formulas for each of the reflections.

If the set is defined by the equation $y_1 = x_1^2$, give the equations for the transformed sets.

Reflections

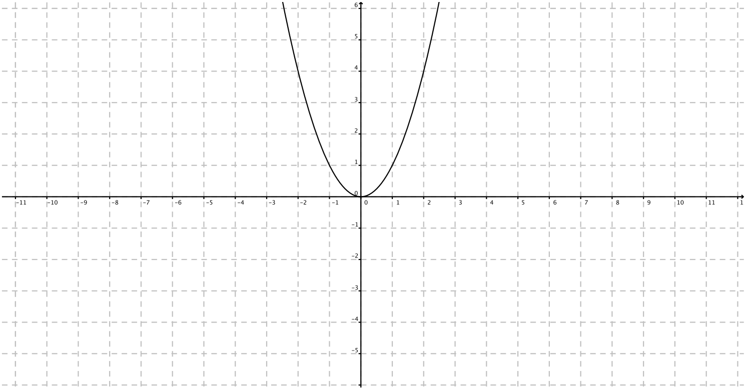
In general, if a set described by an equation in x_1 and y_1 is reflected across one of the previous lines, how do you get the equation of the transformed set?

Reflections

What can you deduce from your answers about the symmetry of the set?

What can you deduce from your answers about finding the inverse of a function described by $y = f(x)$?

Rotations



1. Rotate the given set 90° counterclockwise about the origin.
2. Rotate the given set 180° counterclockwise about the origin.

Rotations

Write formulas for each of the reflections.

If the set is defined by the equation $y_1 = x_1^2$, give the equations for the transformed sets.

Rotations

In general, if a set described by an equation in x_1 and y_1 is rotated counterclockwise by one of the above angles, how do you get the equation of the transformed set?

Rotations

Now try this: What is the resulting equation if the ellipse described by

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

is rotated counterclockwise 45° about the origin?

Completing the Square

Consider the parabola given by the equation $y_1 = 2x_1^2 - 12x_1 + 23$. How can we translate it so that the vertex is at the origin? If the translation is given by $x_2 = x_1 + h$ and $y_2 = y_1 + k$, then we have

$$y_2 - k = 2(x_2 - h)^2 - 12(x_2 - h) + 23$$

or

$$y_2 = 2x_2^2 + (-4h - 12)x_2 + 2h^2 + 12h + 23 + k.$$

We want $-4h - 12$ to be zero, so $h = -3$.
We also want $2h^2 + 12h + 23 + k = 0$ so $k = -5$.
Then $y_2 = 2x_2^2$ is the equation of the translated parabola.
So the equation of the original parabola is $y_1 - 5 = 2(x_1 - 3)^2$ which has vertex $(3, 5)$.

More Rotations

Problem from my High School course:

What is the resulting equation if the parabola described by $y_1 = x_1^2$ is rotated counterclockwise about the origin by the angle t having $\sin t = \frac{7}{25}$ and $\cos t = \frac{24}{25}$?

We use the rotation formulas derived earlier:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}.$$

Substituting, we have

$$-\frac{7}{25}x_2 + \frac{24}{25}y_2 = \left(\frac{24}{25}x_2 + \frac{7}{25}y_2\right)^2$$

which simplifies to

$$576x^2 + 336xy + 49y^2 + 175x - 600y = 0.$$

Conics

Problem from my High School course:
Analyze the conic given by the equation

$$73x^2 - 72xy + 52y^2 - 410x + 120y + 525 = 0.$$

We wish to apply a rotation by angle t that eliminates the xy term. We use the rotation formulas derived earlier (with $x_1 = x$ and $y_1 = y$). Let's abbreviate $s = \sin t$ and $c = \cos t$.

$$\begin{aligned} x &= cx_2 + sy_2, \\ y &= -sx_2 + cy_2. \end{aligned}$$

After substitution and simplification we find that the coefficient of x_2y_2 is

$$42sc - 72(c^2 - s^2).$$

Conics

We need an angle t so that this expression equals 0. Let $T = 2t$, $S = \sin T$, and $C = \cos T$. Then $S = 2sc$ and $C = c^2 - s^2$ by the Double Angle Formulas. So we want an angle T with

$$21S - 72C = 0.$$

But this means $\tan T = \frac{S}{C} = \frac{24}{7}$. From this (and the Pythagorean Theorem) we calculate $S = \frac{24}{25}$ and $C = \frac{7}{25}$.

Conics

Now we use the Half Angle Formulas to find s and c:

$$s = \sqrt{\frac{1-C}{2}} = \frac{3}{5},$$

$$c = \sqrt{\frac{1+C}{2}} = \frac{4}{5}.$$

Using these values of c and s, the rotated conic has equation

$$100x_2^2 - 400x_2 + 25y_2^2 - 150y_2 + 525 = 0.$$

Conics

Complete the two squares to get

$$100(x_2^2 - 4x_2 + 4) + 25(y_2^2 - 6y_2 + 9) = 100,$$

or

$$(x-2)^2 + \frac{(y-3)^2}{4} = 1.$$

This is an ellipse with center $(2,3)$.

Conics

So we have deduced that the original ellipse can be obtained from the ellipse $x^2 + \frac{y^2}{4} = 1$ by first translating it by $(2, 3)$ and then rotating it clockwise by the angle t with $\sin t = \frac{3}{5}$ and $\cos t = \frac{4}{5}$.

Modeling

Motivations

1. *Common Core State Standards*
2. *Essential Understandings*

Modeling

Common Core State Standards
Mathematical Practice 4: Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later.

Modeling

They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

Modeling

Big Idea 3 from *Developing Essential Understandings of Functions Grades 9–12*:

Functions can be classified into different families of functions, each with its own unique characteristics. Different families can be used to model different real-world phenomena.

Fountains

Start GeoGebra and use the "Insert Image" command to paste a picture in the background of an arc of water, such as http://farm4.static.flickr.com/3258/2847369024_be4a45303a.jpg. Try entering, and then modifying (e.g., via the Algebra window, or by the various geometric transformations, such as translate and dilate—try using a slider), a function whose graph is a parabola, to match the shape of the arcs. You might try $f(x) = a(b(x-c))^2 + d$ where a, b, c, and d are determined by sliders. Why is a parabola appropriate?

Bouncing balls also follow parabolic paths — find a stop-action photo to try, such as

http://people.rit.edu/andpph/photofile-sci/bouncing-ball-7810.jpg.

St. Louis Arch

This time try to match an image of the St. Louis arch, which is a catenary, having an equation of the form $y = \frac{a}{2}(e^{x/a} + e^{-x/a})$.
Here is an image:
`http://3547.voxcdn.com/photos/0/34/46457_l.jpg`.

Vibrating String

Try to match an image of a vibrating string with a sine function. Here is an image, but you have to rotate it 90 degrees with some other software before bringing it into GeoGebra:

http://people.rit.edu/andpph/text-figures/strings/vibration-setup-8181a.jpg.

Nautilus

Try to match an image of the nautilus with a logarithmic spiral. Here is an image:

http://www.nassmc.org/about_images/nautilus.jpg.

For the logarithmic spiral, given by $r = ae^{b\theta}$, make sliders for a and b and use the command

Curve$[a * e^{(bt)} * \cos(t), a * e^{(bt)} * \sin(t), t, 0, 25]$. Why does this create a logarithmic spiral?

Sunflower

Now try to match a spiral to a sunflower. Here is an image:
`http://farm2.static.flickr.com/1278/694780262_8874b4f225.jpg`.
How do you reverse the direction of the spiral?

Curve Fitting

Go to the Data and Story Library (DASL), `http://lib.stat.cmu.edu/DASL/DataArchive.html`. Select "List All Topics", then "Health", then "Smoking and Cancer". Select the Datafile. Highlight the six columns of the data set (omit the headers) and copy.

Curve Fitting

Now open GeoGebra and select "Spreadsheet View" under "View". Click on cell A1. Then paste the date in. Highlight the last three columns and press Delete to clear these cells. In cell D1 type "=(B1,C1)". This creates a point labeled D1, which will show up in the graphics view. Highlight cell D1. Then grab the lower right-hand corner and drag it down to create points D1 through D38. All of these points will show up in the graphics view. You may want to right-click the background of the graphics view to adjust the minimum and maximum values of the axes in order to see the points more clearly.

Curve Fitting

To find the line of best fit, enter the following at the bottom of the graphics view: FitLine[D1:D38]. To find the correlation coefficient, enter CorrelationCoefficient[D1:D38].
Now experiment with other data sets, and other "Fit" Commands, like FitPoly[D1:D38,2] or FitPoly[D1:D38,3].

Circular Motion

Let's use trig functions make a simplified model of a planet rotating on its axis while revolving around its sun.

Start in GeoGebra by drawing a unit circle centered at the origin, to represent the planet's orbit. Make a slider, say a, with min and max values 0 and 6.28, respectively. To place a point, say B, on this orbit (for the planet's center), enter $(\cos(t), \sin(t))$.

You can animate the slider by double-clicking on it, selecting "Animation On" under Basic, and "Increasing" under Slider. A small "play/pause" icon appears in the lower left of the view.

Circular Motion

Now we will create a planet by pausing the animation and drawing a circle with center B and radius 0.1. You can then select the circle and fill its interior with a color if you wish.

Place a reference point, say, C, on the planet by entering $B + .1 * (\cos(a), \sin(a))$.

Turn the animation back on. You may wish to pause and adjust the speed and the increment of the slider for smoother motion.

Circular Motion

Notice that right now, the planet rotates exactly once about its axis while it revolves once about the sun. The net effect for the inhabitants of the planet is that the sun does not move across the sky! (This is what our moon is doing with respect to the Earth.)

We can change this by changing the definition of C to be $B + .1 * (\cos(3a), \sin(3a))$. Now the planet rotates around its axis three times during the year. Yet the inhabitants experience only two days — why is that? Experiment with other numbers of days in the year.

Can you add a moon?

Some Technology Resources

- Mathematics Education and Outreach Resources, http://www.ms.uky.edu/~lee/outreach/outreach.html. My master list of useful resources.
- GeoGebra, http://www.geogebra.org, for geometric and algebraic investigations.
- GeoGebra 5.0, http://www.geogebra.org/en/wiki/index.php/Release_Notes_GeoGebra_5.0, a beta version of GeoGebra including 3D features.
- Google SketchUp, http://sketchup.google.com, for three-dimensional modeling.
- WolframAlpha, http://www.wolframalpha.com, for online calculation, including graphing, symbolic algebra.
- Wolfram Demonstrations Project, http://demonstrations.wolfram.com, for many illustrative applets.

Some Technology Resources

- National Library of Virtual Manipulatives, `http://nlvm.usu.edu`, for K–12 applets.
- NCTM's Illuminations, `http://illuminations.nctm.org`, for activities and lessons.
- Interactivate, `http://www.shodor.org/interactivate`, for activities and lessons.
- Data and Story Library, `http://lib.stat.cmu.edu/DASL`, for many interesting data sets.
- Common Core State Standards for Mathematics, `http://www.corestandards.org/the-standards`.
- Jing, `http://www.techsmith.com/Jing`, for capturing screen images and making videos.
- Dropbox, `http://www.dropbox.com`, for storing and sharing files with others.