Squeezed 2-Spheres and 3-Spheres are Hamiltonian

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1 Introduction

A (convex) d-polytope is said to be *Hamiltonian* if its edge-graph is. It is well-known that simple 3-polytopes are not, in general, Hamiltonian, though non-Hamiltonian examples require at least 38 vertices [3, 4]. The question of whether all simple d-polytopes, $d > 3$, are Hamiltonian or not remains unsettled. Barnette (see [8]) conjectures that all simple 4-polytopes are Hamiltonian. In partial support of this conjecture, simple 4-prisms over 4-colorable (and hence all) 3-polytopes [8], and duals of cyclic d-polytopes [2, Section 17.2] are Hamiltonian.

The duals of simple d-polytopes are simplicial d-polytopes, and their boundary complexes are therefore simplicial $(d-1)$-spheres. Thus it is natural to say that a pure simplicial $(d-1)$-complex has a *Hamiltonian path* if its facets (maximal faces) can be ordered $F_1, \ldots, F_m$ such that F_i and F_{i+1} are adjacent (intersect in a set of cardinality $d-1$) for all $1 \leq i \leq m-1$. If, further, F_1 and F_m are also adjacent, then we say that the simplicial complex has a *Hamiltonian cycle* or is *Hamiltonian.*

In this paper we prove that all squeezed 2- and 3-spheres are Hamiltonian. In creating the collection of squeezed $(d-1)$-spheres, Kalai [5] establishes a large lower bound on the number of labeled simplicial $(d-1)$-spheres, and hence a substantial gap between the number of spheres and the number of simplicial convex d-polytopes. This collection extends the construction of [1] and thus contains at least one polytopal representative for every possible simplicial d-polytope f-vector. As a consequence, should there be a non-Hamiltonian simple 4-polytope, it cannot be detected by its f-vector alone.

2 Definitions

We begin by reviewing the construction and some basic properties of squeezed balls and spheres [5, 7].

For an integer $n \geq 1$, let $[n]$ denote the set $\{1, \ldots, n\}$. For a nonnegative integer d, $[n]^{(d)}$ is the set of all subsets of $[n]$ of cardinality d. For $F \in [n]^{(d)}$, we will write $F - i$ for $F \setminus \{i\}$ and $F + i$ for $F \cup \{i\}$. For $C \subseteq 2^{[n]}$, define $\mathrm{cone}(C, 0) = \{F + 0 : F \in C\}$.

Let $F, G \in [n]^{(d)}, F = \{a_1, \ldots, a_d\}, G = \{b_1, \ldots, b_d\}$, where $a_1 < \cdots < a_d$ and $b_1 < \cdots < b_d$. We say $F \leq_p G$ if $a_i \leq b_i$ for $1 \leq i \leq d$. Define $F \leq_L G$ if $\min(F \bigtriangleup G) \in F$ where $F \bigtriangleup G$ is the symmetric difference of the sets F and G. A collection ordered with respect to $\leq_L$ is said to be in *lexicographic order.*

The *reverse lexicographic order* on $[n]^{(d)}$ is defined by $F <_{RL} G$ if $\max(F \bigtriangleup G) \in G$. Define $q(F, G) = \min\{j : a_i = b_i \text{ for all } i \geq j\}$ taking $q(F, G) = d + 1$ if $a_d \neq b_d$.

Now suppose F is a proper subset of $[n]$. Let $j = \min\{i \in [n] : i \notin F\}$ and $k = \max\{i \in [n] : i \notin F\}$. We call $\{1, \ldots, j - 1\}$ the *left* set of F and $\{k + 1, \ldots, n\}$ the *right* set of F. If $s, t \notin F$, $s, t \in [n]$, $s < t$, but $i \in F$ for all $s + 1 \leq i \leq t - 1$, then $\{s + 1, \ldots, t - 1\}$ is a *middle* set of F. Note that left, right, or middle sets may be empty.

Suppose $j \in F \subseteq [n]$. If j is not in the left set of F, define $\ell(F, j) = \max\{i \in [n] : i < j, i \notin F\}$ and $L(F, j) = (F - j) + \ell(F, j)$. If j is not in the right set of F, define $r(F, j) = \min\{i \in [n] : i > j, i \notin F\}$ and $R(F, j) = (F - j) + r(F, j)$. These definitions correspond to "pushing" a contiguous set of elements of F to the "left" or "right" respectively. For example if F is the set $\{1, 2, 4, 5, 6, 7\}$, then $L(F, 5)$ is the set $\{1, 2, 3, 4, 6, 7\}$ and $R(F, 4)$ is the set $\{1, 2, 5, 6, 7, 8\}$.

Let d be a positive odd integer. Define $F_d(n)$ to be the collection of all members of $[n]^{d+1}$ having even cardinality left, right, and middle sets. Note that for $F \in F_d(N)$, if we write $F = \{a_1, \ldots, a_{d+1}\}$, we implicitly assume that the elements have been indexed so that $a_1 < \cdots < a_{d+1}$. For even positive d, let $F_d(n) = \mathrm{cone}(F_{d-1}, 0)$.

For positive d, consider a nonempty subcollection, I, of $F_d[n]$ that is an initial set of $F_d[n]$ with respect to the partial order $\leq_p$. Equivalently, if $F \in I$, then $L(F, j) \in I$ for every j for which $L(F, j) \in F_d[n]$. Kalai [5] proves that the simplicial d-complex $B(I) = \{G : G \subseteq F$ for some $F \in I\}$ is topologically a d-ball. He calls such a simplicial complex a *squeezed ball.* Note that the facets of $B(I)$ are precisely the sets contained in I. Kalai observes that squeezed d-balls for even d are just those simplicial complexes of the form $\mathrm{cone}(B(J), 0)$, where J is some initial set with respect to $\leq_p$ of $F_{d-1}(n)$.

The boundary $S(I) = \partial B(I)$ of a squeezed ball $B(I)$ is topologically a $(d - 1)$-sphere, and is called a *squeezed sphere.* The facets of $S(I)$ are those subsets of $B(I)$ of cardinality d that are contained in exactly one facet of $B(I)$.

Kalai proves that the reverse lexicographic ordering of the facets of a squeezed ball $B(I)$ is a shelling order. By examining this shelling, one can readily characterize the facets of $S(I)$ [7].

Proposition 2.1 *Suppose d is odd and $B(I)$ is a squeezed d-ball.*

Let $F = \{a_1, \ldots, a_{d+1}\} \in I$ be a facet of $B(I)$ and $a_i \in F$. Then $F - a_i$ is a facet of $S(I)$ if and only if:

1. *i is even and a_i is in the left set of F, or*

2. *i is odd and $R(F, a_i) \notin I$.*

Suppose d is even and $B(I) =$cone$(B(J), 0)$ is a squeezed d-ball. Then $F \in B(I)$ of cardinality d is a facet of $S(I)$ if and only if:

1. *F is a facet of $B(J)$, or*

2. *$F = G + 0$ where G is a facet of $S(J)$.*

3 Squeezed 2-Spheres are Hamiltonian

Let S be any simplicial $(d-1)$-complex with vertex set V, F be a facet of S, and $v \notin V$. The simplicial complex resulting from the *stellar subdivision* of F is $(S \setminus \{F\}) \cup \{G + v : G \subset F, G \neq F\}$. A simplicial $(d-1)$-complex is a *stacked sphere* if it can be obtained from the boundary of a d-simplex, $\{G \subset H : G \neq H\}$ (where H has cardinality $d+1$), by a sequence of stellar subdivisions of facets. Stacked spheres are all polytopal, and are precisely the boundary complexes of stacked polytopes. Stacked polytopes are dual to truncation polytopes, those obtained from a d-simplex by a sequence of vertex truncations.

An easy proof by induction shows that stacked spheres are Hamiltonian (and in fact stacked 2-spheres admit precisely three Hamiltonian cycles [6]). We will show that squeezed 2-spheres are Hamiltonian by showing that they are stacked.

In this section we use the notation v_i instead of just i when referring to a vertex of a squeezed ball or sphere. We will also abbreviate a set $\{v_i, v_j, v_k, v_\ell\}$ by $v_iv_jv_kv_\ell$, with the convention that $i < j < k < \ell$.

Theorem 3.1 *Squeezed 2-spheres are stacked.*

Proof. Let B be a squeezed 3-ball with vertex set $\{v_1, \ldots, v_n\}$. Its facets are faces of cardinality 4, and its subfacets are faces of cardinality 3. Let ∂B denote the boundary complex of B. The facets of ∂B are those subfacets of B that are contained in exactly one facet of B.

We will prove the theorem by induction on the number of vertices of ∂B. Note that if ∂B is the boundary of a simplex, we are done.

We say a facet F of B is of *type* j if $j = \min\{i : v_i \in F\}$. See, for example, the 3-ball in Figure 1, in which we abbreviate v_i as i. Let k be the maximum type of a facet of B.

	F_1	1	2	3	4				
	F_2	1	2		4	5			
	F_3	1	2			5	6		
	F_4	1	2				6	7	
	F_5	1	2					7	8
•	F_6		2	3	4	5			
•	F_7		2	3		5	6		
•	F_8		2	3			6	7	
•	F_9		2	3				7	8
	F_{10}			3	4	5	6		
	F_{11}			3	4		6	7	

Figure 1: The facets of a squeezed 3-ball, with its type 2 facets indicated by •

A useful consequence of the fact that B is squeezed is that every set of the form $v_i v_{i+1} v_j v_{j+1}$ where $j \leq k+2$ is a facet of B.

Case I: Assume $k = 1$. By the definition of a squeezed ball, the facets of B must be of the form $v_1v_2v_3v_4, v_1v_2v_4v_5, \ldots, v_1v_2v_{n-1}v_n$. Thus they each contain vertices 1 and 2, and each facet of the form $v_1v_2v_{j-1}v_j$ meets the facet $v_1v_2v_{j-2}, v_{j-1}$ in the common face $v_1v_2v_{j-1}$, $5 \leq j \leq n$. Starting with $v_1v_2v_3v_4$, with the addition of each facet $v_1v_2v_{j-1}v_j$, $5 \leq j \leq n$, ∂B changes by a stellar subdivision of the sphere facet $v_1v_2v_{j-1}$. Therefore ∂B is a stacked sphere.

Case II: Suppose $k \geq 2$. Let F be the lexicographically-least facet of type k. So $F = v_k v_{k+1} v_{k+2} v_{k+3}$. Note that no other facets of type k contain v_{k+2}. For example, in Figure 1, $k = 3$, $F = F_{10}$, and $v_{k+2} = v_5$.

We first show that v_{k+2} is contained in exactly three facets of ∂B. For $1 \leq j \leq k-1$ let $A_j = v_j v_{j+1} v_{k+2} v_{k+3}$ and $B_j = v_j v_{j+1} v_{k+1} v_{k+2}$. Both A_j and B_j are facets of B since B

is squeezed. The set of facets of B containing v_{k+2} is $\{F, A_1, B_1, \ldots, A_{k-1}, B_{k-1}\}$. Now, for $1 \leq j \leq k-1$, we have $v_j v_{j+1} v_{k+2} = A_j \cap B_j$. So $v_j v_{j+1} v_{k+2} \notin \partial B$. For $2 \leq j \leq k-1$, $v_j v_{k+1} v_{k+2} = B_j \cap B_{j-1}$. Also, for the same j, we have $v_j v_{k+2} v_{k+3} = A_j \cap A_{j-1}$. Thus $v_j v_{k+1} v_{k+2}, v_j v_{k+2} v_{k+3} \notin \partial B$ for $2 \leq j \leq k-1$. Finally, $v_k v_{k+2} v_{k+3} = A_{k-1} \cap F$, and $v_k v_{k+1} v_{k+2} = B_{k-1} \cap F$. So $v_k v_{k+1} v_{k+2}, v_k v_{k+2} v_{k+3} \notin \partial B$. This leaves v_{k+2} only in the three facets $v_1 v_{k+1} v_{k+2}$, $v_1 v_{k+2} v_{k+3}$ and $v_{k+1} v_{k+2} v_{k+3}$ of ∂B.

Next, we construct a new simplicial complex from the ball B. Its facets are obtained from the facets of B by removing F and, for each $1 \leq j \leq k-1$, replacing A_j and B_j with $C_j = v_j v_{j+1} v_{k+1} v_{k+3}$. This is a squeezed ball B' with vertices $v'_1, \ldots, v'_{n-1}$ where $v'_j = v_j$ for $1 \leq j \leq k+1$ and $v'_j = v_{j+1}$ for $k+2 \leq j \leq n-1$. See the example in Figure 2.

B

	1	2	3	4				
B_1	1	2		4	5			
A_1	1	2			5	6		
	1	2				6	7	
	1	2					7	8
B_2		2	3	4	5			
A_2		2	3		5	6		
		2	3			6	7	
		2	3				7	8
F			3	4	5	6		
			3	4		6	7	

B'

	1	2	3	4			
C_1	1	2		4	6		
	1	2			6	7	
	1	2				7	8
C_2		2	3	4	6		
		2	3		6	7	
		2	3			7	8
			3	4	6	7	

Figure 2: The facets of B and B'

The collection $\mathcal{F}(\partial B)$ of facets of ∂B not containing v_{k+2} is identical to the collection $\mathcal{F}(\partial B')$ of facets of $\partial B'$ apart from $v_1 v_{k+1} v_{k+3}$. This correspondence is detailed in Figure 3.

Case 1: Suppose $G = v_i v_{i+1} v_j v_{j+1} \in B$ and $G - v_i$ is a facet of ∂B not containing v_{k+2}. Then $R(G, v_i) \notin B$ and necessarily $j > k+2$ (using the fact that B is squeezed). So $G \notin \{F, A_1, B_1, \ldots, A_{k-1}, B_{k-1}\}$ and $R(G, v_i) \notin \{C_1, \ldots, C_{k-1}\}$. Therefore $G \in B'$, $R(G, v_i) \notin B'$, and $G - v_i \in \partial B'$.

Case 2: Suppose $G = v_1 v_2 v_j v_{j+1} \in B$ and $G - v_2$ is a facet of ∂B not containing v_{k+2}. Then $G \notin \{F, A_1, B_1, \ldots, A_{k-1}, B_{k-1}\}$. Therefore $G \in B'$ and $G - v_2 \in \partial B'$.

	$\mathcal{F}(\partial B)$	$\mathcal{F}(\partial B')$
Case 1:	$v_i v_{i+1} v_j v_{j+1} - v_i$	$v_i v_{i+1} v_j v_{j+1} - v_i$
Case 2:	$v_1 v_2 v_j v_{j+1} - v_2$	$v_1 v_2 v_j v_{j+1} - v_2$
Case 3a:	$v_k v_{k+1} v_{k+2} v_{k+3} - v_{k+2}$	$v_{k-1} v_k v_{k+1} v_{k+3} - v_{k-1}$
Case 3b:	$v_i v_{i+1} v_{k+2} v_{k+3} - v_{k+2}$ $(i < k)$	$v_i v_k v_{i+1} v_{k+1} v_{k+3} - v_{k+1}$
Case 3c:	$v_i v_{i+1} v_j v_{j+1} - v_j$ $(j \neq k+2)$	$v_i v_{i+1} v_j v_{j+1} - v_j$
Case 4:	$v_1 v_2 v_3 v_4 - v_4$	$v_1 v_2 v_3 v_4 - v_4$

Figure 3: Correspondence between $\mathcal{F}(\partial B)$ and $\mathcal{F}(\partial B')$

Case 3a: Suppose $G = v_k v_{k+1} v_{k+2} v_{k+3} \in B$ and $G - v_{k+2}$ is a facet of ∂B not containing v_{k+2}. Then $G = F$ and $H = v_k v_{k+1} v_{k+3} v_{k+4} = R(G, v_{k+2}) \notin B$. Now $C_{k-1} = v_{k-1} v_k v_{k+1} v_{k+3} \in B'$ and $R(C_{k-1}, v_{k-1})$ (with respect to B') is $H \notin B'$. Therefore $C_{k-1} - v_{k-1} \in \partial B'$.

Case 3b: Suppose $G = v_i v_{i+1} v_{k+2} v_{k+3} \in B$ and $G - v_{k+2}$ is a facet of ∂B not containing v_{k+2}, where $i < k$. Then $G = A_i$ and $H = v_i v_{i+1} v_{k+3} v_{k+4} = R(G, v_{k+2}) \notin B$. Now $C_i = v_i v_{i+1} v_{k+1} v_{k+3} \in B'$ and $R(C_i, v_{k+1})$ (with respect to B') is $H \notin B'$. Therefore $C_i - v_{k+1} \in \partial B'$.

Case 3c: Suppose $G = v_i v_{i+1} v_j v_{j+1} \in B$ and $G - v_j$ is a facet of ∂B not containing v_{k+2}, where $j \neq k+2$. Obviously $v_{k+2} \notin G - v_j$ implies $j \neq k+1$ either. So $G \notin \{F, A_1, B_1, \ldots, A_{k-1}, B_{k-1}\}$. Also $R(G, v_j) \notin B$, so necessarily $j > k+2$ (using the fact that B is squeezed). So $R(G, v_j) \notin \{C_1, \ldots, C_{k-1}\}$. Therefore $G \in B'$, $R(G, v_j) \notin B'$, and $G - v_j \in \partial B'$.

Case 4: Suppose $G = v_1 v_2 v_3 v_4$. Then $G \in B$, $G \in B'$, $G - v_4 \in \partial B$ and $G - v_4 \in \partial B'$.

It is straightforward to check that all possibilities of facets in $\mathcal{F}(\partial B')$ have been accounted for, establishing the correspondence between $\mathcal{F}(\partial B)$ and $\mathcal{F}(\partial B')$. This analysis implies that ∂B is obtained from $\partial B'$ by the stellar subdivision of the facet $v_1 v_{k+1} v_{k+3}$. Now $\partial B'$ is a squeezed 2-sphere with one fewer vertex than ∂B, and by the induction hypothesis $\partial B'$ is stacked. Therefore ∂B is stacked and hence, by induction, squeezed 2-spheres are stacked. □

Corollary 3.2 *Squeezed 2-spheres are Hamiltonian.*

4 Squeezed 3-Spheres are Hamiltonian

In this section we show every squeezed 3-sphere is Hamiltonian by exhibiting an explicit ordering of its facets. We first show that the shelling order of the facets of a squeezed 2-sphere described in [7] is actually a Hamiltonian path. We then use this and the fact that a squeezed 4-ball B is a squeezed 3-ball joined to the point 0 to get a path involving the facets ∂B containing 0. Then we insert the remaining facets of ∂B to form a Hamiltonian cycle for ∂B.

The shelling order for squeezed $(d-1)$-spheres in [7] specializes to boundaries of squeezed 3-balls B in the following way. Let $F = a_1a_2a_3a_4$ be a facet of B. (Implicit in this notation is that $F = \{a_1, a_2, a_3, a_3\}$ where $a_1 < a_2 < a_3 < a_4$.) Suppose $F - a_k \in \partial B$.

The only such facets of ∂B when k is even are $123 = 1234 - 4$ and $12a_3a_4 - 2$. The shelling order for ∂B begins by ordering these "even" facets lexicographically. In Figure 4 we list the facets of a 3-ball B and the ordering of the facets of ∂B of the form $F - a_k$ where k is even.

B

1	2^2	3	4^1					
1	2^3		4	5				
	2	3	4	5				
1	2^4			5	6			
	2	3		5	6			
		3	4	5	6			
1	2^5				6	7		
	2	3			6	7		
		3	4		6	7		
1	2^6					7	8	
	2	3				7	8	
1	2^7						8	9
	2	3					8	9

∂B

H_1	1	2	3						
H_2	1		3	4					
H_3	1			4	5				
H_4	1				5	6			
H_5	1					6	7		
H_6	1						7	8	
H_7	1							8	9

Figure 4: A squeezed 3-ball B and the ordering of the "even" facets of ∂B

The shelling order then continues with the boundary facets of the form $F - a_k$ where k is odd in the following manner. Suppose the facets of B in reverse lexicographic order are $F_1, F_2, \ldots, F_m$. Define $\mathcal{G}_p = \{F_k, F_{k+1}, \ldots, F_\ell\}$ to be those facets for which $a_4 = p$. In each of these groups $\mathcal{G}_p$, find the minimum i so that $F_i - a_3 \in \partial B$ is a boundary facet and order the "odd" facets of ∂B associated with this group as $F_i - a_3, F_{i+1} - a_3, \ldots, F_{\ell-1} - a_3, F_\ell - a_3, F_\ell - a_1$. Let $\mathcal{G}_p^*$ be this ordered list of these "odd" facets of ∂B. Note that it is possible that $\mathcal{G}_p^*$ is empty (but not if $p = n$, the number of vertices of B), and that if $\mathcal{G}_p^*$ has only one facet, it is $F_\ell - a_1$. Now order all of the "odd" facets of ∂B by listing them in the order $\mathcal{G}_n^*, \mathcal{G}_{n-1}^*, \mathcal{G}_{n-2}^*, \ldots$ and so on until $\mathcal{G}_{n-t}^* = \emptyset$. It is a fact that if $\mathcal{G}_p^* = \emptyset$ then $\mathcal{G}_i^* = \emptyset$ for all $i < p$. In Figure 5, we list the facets of a 3-ball B and the ordering of the facets of ∂B the form $F - a_k$ where k is odd.

B

1	2	3	4					
1	2		4	5				
	2	3	4	5				
1	2			5	6			
	2	3		5	6			
		3^{14}	4	5	6			
1	2				6	7		
	2	3			6	7		
		3^{13}	4		6^{12}	7		
1	2					7	8	
	2^{11}	3				7	8	
1	2						8^8	9
	2^{10}	3					8^9	9

∂B

	1	2	3						
	1		3	4					
	1			4	5				
	1				5	6			
	1					6	7		
	1						7	8	
	1							8	9
H_8	1	2							9
H_9		2	3						9
H_{10}			3					8	9
H_{11}			3				7	8	
H_{12}			3	4			7		
H_{13}				4		6	7		
H_{14}				4	5	6			

Figure 5: A squeezed 3-ball B and the ordering of the seven remaining "odd" facets of ∂B

Proposition 4.1 *The above shelling order of squeezed 2-spheres is a Hamiltonian path.*

Proof. Let $F - a_k$ be a facet of ∂B where $F \in B$. We will show, for each possibility of k and F, which adjacent facet $G - b$ immediately precedes $F - a_k$ in the shelling order.

Suppose k is even. If $F = 1234$ and $k = 4$, then $F - a_4 = F - 4$ is the first facet in the shelling order. If $F = 1234$ and $k = 2$, then $F - 2$ is the second facet in the shelling order, and is adjacent to $F - 4$. Otherwise, since k is even, $F = 12a_3a_4$, $a_3 > 3$, and $k = 2$. Let $G = L(F, a_4)$. Now $G - 2$ is adjacent to $F - 2$ and it is evident from the shelling order that $G - 2$ is the immediate predecessor of $F - 2$.

Suppose k is odd.

Case I: $k = 1$. That is, we are considering a facet of the form $F - a_1$. Now either $F - a_3$ is a facet of ∂B or it is not a facet.

If $F - a_3$ is a facet of ∂B, then $F - a_3$ and $F - a_1$ are adjacent, and it is evident from the shelling order that $F - a_3$ is the immediate predecessor of $F - a_1$. Otherwise, $F - a_3$ is not a facet of ∂B. Thus $G = R(F, a_3) \in B$. Now $F - a_1 \in \partial B$ implies that $R(F, a_1) \notin B$, which implies that $R(G, a_1) = R(R(F, a_3), a_1) \notin B$ since B is squeezed. Thus $G - a_1 = R(F, a_3) - a_1$ is a facet of ∂B. Now $G - a_1$ is adjacent to $F - a_1$, and it is evident from the shelling order that $G - a_1$ is the immediate predecessor of $F - a_1$.

Case II: $k = 3$. That is, we are considering a facet of ∂B of the form $F - a_3$ where $F \in B$. Recall that $F - a_3$ a facet of ∂B implies that $R(F, a_3) \notin B$. Now either $F = 12a_3a_4$ or $L(F, a_2) \in B$ (since B is squeezed).

If $F = 12a_3a_4$, then $R(F, a_3) \notin B$ forces $a_3 = n - 1$ and $a_4 = n$. So $F - a_3$ is the first "odd" facet of ∂B in the shelling order. Thus $F - 2$ immediately precedes $F - a_3$ in the shelling order, and it is also adjacent.

Now suppose $L(F, a_2) \in B$.

Subcase 1: $G = L(F, a_2)$ and $G - a_3$ is a facet of ∂B. Easily $G - a_3$ is adjacent to $F - a_3$, and in the shelling order $G - a_3$ is the immediate predecessor of $F - a_3$.

Subcase 2: $L(F, a_2) - a_3$ is not a facet of ∂B. Thus $R(L(F, a_2), a_3) \in B$. Let b be the smallest element of G, where $G = R(L(F, a_2), a_3)$. Now $G - b$ is a facet of ∂B since $R(R(L(F, a_2), a_3), b) = R(F, a_3) \notin B$. Also, $G - b$ is adjacent to $F - a_3$, and $G - b$ is the immediate predecessor of $F - a_3$.

Thus the shelling order is Hamiltonian path. $\square$

We are now ready to construct Hamiltonian cycles for squeezed 3-spheres.

Theorem 4.2 *Squeezed 3-spheres are Hamiltonian.*

Proof. Let B be a squeezed 4-ball and ∂B be the associated squeezed 3-sphere. Now $B = \text{cone}(B', 0)$ for some squeezed 3-ball B'. The above shelling order for $\partial B'$ induces an ordering of the facets of ∂B containing 0 that is a Hamiltonian path. The last facet of this path is a facet of the form $F - a_1$ where

- $F \in B$,

- $F = 0a_1a_2a_3a_4$ where $a_j = a_1 + (j-1)$ for $j = 2, 3, 4$, and
- there does not exist a facet G of the 4-ball satisfying the above two conditions for which $F <_{RL} G$ (otherwise, $F - a_1$ would not be a facet).

See Figure 6.

	0	1	2	3	4					
	0	1	2		4	5				
	0		2	3	4	5				
	0	1	2			5	6			
	0		2	3		5	6			
F	0			3	4	5	6			
	0	1	2				6	7		
	0		2	3			6	7		
	0			3	4		6	7		
	0	1	2					7	8	
	0		2	3				7	8	
	0	1	2						8	9
	0		2	3					8	9

Figure 6: The facets of a squeezed 4-ball

Let $S_0 = \{G - a_i : G \in B,\ G - a_i \in \partial B \text{ and } 0 \in G - a_i\}$, $S_1 = \{G - 0 : G \in B \text{ and } G \leq_{RL} F\}$, and $S_2 = \{G - 0 : G \in B \text{ and } F <_{RL} G\}$.

Order the facets in S_0 using the reverse of ordering induced by the shelling of $\partial B'$. Note that this ordering begins with the facet $F - a_1$ and ends with the facet 0123. Let $S_1 = S_{14} \cup S_{15} \cup \ldots \cup S_{1a_4}$ where $S_{1k} = \{G : G \in S_1 \text{ and } k = \max G\}$. Order the facets within each S_{1k} from lexicographically greatest to lexicographically least if $a_4 - k$ is an odd number and from lexicographically least to lexicographically greatest if $a_4 - k$ is an even number. Concatenate these orderings in the sequence $S_{14}, S_{15}, \ldots, S_{1a_4}$. This results in a Hamiltonian path of the facets in S_1 that begins with the facet 1234 and ends with the facet $F - 0$. Note that in the example of Figure 7 we have S_0 consisting of facets F_1 through F_{14}, S_1 consisting of facets F_{15} through F_{20}, S_{14} consisting of the facet F_{15}, S_{15} consisting of facets F_{16} through F_{17}, S_{16} consisting of facets F_{18} through F_{20}, and $a_4 = 6$.

F_1	0				4	5	6			
F_2	0				4		6	7		
F_3	0			3	4			7		
F_4	0			3				7	8	
F_5	0			3					8	9
F_6	0		2	3						9
F_7	0	1	2							9
F_8	0	1							8	9
F_9	0	1						7	8	
F_{10}	0	1					6	7		
F_{11}	0	1				5	6			
F_{12}	0	1			4	5				
F_{13}	0	1		3	4					
F_{14}	0	1	2	3						
F_{15}		1	2	3	4					
F_{16}			2	3	4	5				
F_{17}		1	2		4	5				
F_{18}		1	2			5	6			
F_{19}			2	3		5	6			
F_{20}				3	4	5	6			

Figure 7: Some of the facets of the boundary of the squeezed 4-ball

Now concatenating these paths forms a cycle, $\mathcal{H}_1$, of the facets in $S_0 \cup S_1$. The remaining facets, S_2, of ∂B will be "inserted" into $\mathcal{H}_1$ carefully maintaining the cycle of facets at each insertion.

Now S_2 consists of facets of the form $Q = b_1b_2b_3b_4$ where $0 < b_1 \leq a_1$ and $a_3 < b_3$. (Otherwise, $Q + 0 \leq_{RL} F$ or $R(F, a_1) \in B$ contradicting the definition of F.) Let $W_i = \{Q = b_1b_2b_3b_4 : Q \in S_2 \text{ and } b_3 = a_3 + i\}$. For each odd i, consider W_i and W_{i+1}. If $W_{i+1} = \emptyset$ then set $W_i' = \emptyset$.

Suppose $W_{i+1} \neq \emptyset$. Let $P_{i+1} = \max_{<_{RL}}\{Q : Q \in W_{i+1}\}$ and $P_i = L(P_{i+1}, b_4) \in W_i$ where $P_{i+1} = b_1b_2b_3b_4$. Let $W_i' = W_{i+1} \cup \{P : P \in W_i \text{ and } P \leq_{RL} P_i\}$. See Figure 8, in which P_2 is $F_5 - 0$ and P_1 is $F_2 - 0$.

Listing the elements of $W_i \cap W_i'$ from least to greatest in reverse lexicographic order, and the remainder of the elements of W_i' from greatest to least in reverse lexicographic order,

	0	1	2	3	4					
	0	1	2		4	5				
	0		2	3	4	5				
	0	1	2			5	6			
	0		2	3		5	6			
F	0			3	4	5	6			
F_1	0	1	2				6	7		
F_2	0		2	3			6	7		
F_3	0			3	4		6	7		
F_4	0	1	2					7	8	
F_5	0		2	3				7	8	
F_6	0	1	2						8	9
F_7	0		2	3					8	9

Figure 8: The 4-ball: $V_1 = \{F_1, F_2, F_3\}$ and $V_2 = V_{1+1} = \{F_4, F_5\}$.

we have $W_i' = \{12bb_3, 23bb_3, \ldots, b_1b_2bb_3 = P_i, b_1b_2b_3b_4 = P_{i+1}, L(P_{i+1}, b_2), \ldots, 12b_3b_4\}$, where $b = b_3 - 1 = \ell(P_{i+1}, b_4)$. Note that the above ordering is a Hamiltonian path of the facets in W_i'.

Now $012bb_3$ and $012b_3b_4 \in B$ and hence $01bb_3$ and $01b_3b_4 \in S_0$. Also $01bb_3$ is adjacent to $01b_3b_4$ in $\mathcal{H}_1$. Insert W_i' in the above order between $01bb_3$ and $01b_3b_4$ in $\mathcal{H}_1$. Do this for each i for which $W_i' \neq \emptyset$ to form a Hamiltonian cycle, $\mathcal{H}_2$, on the facets in $S_0 \cup S_1 \cup \{P : P \in W_i'$ for some $i\}$.

Now we must explain what to do with the facets in $S_2 - \cup_{i \text{ odd}} W_i'$. Let P belong to $S_2 - \cup_{i \text{ odd}} W_i'$. Thus $P \in W_i$ for some odd i, and $P_i <_{RL} P$ if $W_{i+1} \neq \emptyset$. Note that $S_2 - \cup_{i \text{ odd}} W_i'$ contains all of W_i if $W_{i+1} = \emptyset$ and i is odd.

Suppose i (odd) is such that $W_{i+1} \neq \emptyset$. Let $Q_1 <_{RL} Q_2 <_{RL} \cdots <_{RL} Q_m$ be the facets in W_i such that $P_i <_{RL} Q_j$ for all j. For each j, let $G_j = 0b_1^jb_2^jb_3b_4$, where $P_i = b_1b_2b_3b_4$, and $Q_j = b_1^jb_2^jb_3b_4$. Now for each j, $G_j - b_3$ is a facet of ∂B since $P_{i+1} = \max_{<_{RL}}\{Q : Q \in W_{i+1}\} <_{RL} R(G_j, b_3) - 0$ and hence $R(G_j, b_3) \notin B$.

Note that for $1 \leq j \leq m-1$, $G_j - b_3$ is adjacent to $G_{j+1} - b_3$. Also $G_m - b_1^m$ is adjacent to $G_m - b_3$ in $\mathcal{H}_2$. For each odd $j < m$ insert $G_j - 0$ and $G_{j+1} - 0$ between $G_j - b_3$ and $G_{j+1} - b_3$ in $\mathcal{H}_2$. Observe that $G_j - 0$ is adjacent to $G_{j+1} - 0$ since $G_j = L(G_{j+1}, b_2^{j+1})$. If m is odd, place $G_m - 0$ between $G_m - b_1^m$ and $G_m - b_3$. Do this for each i for which $W_{i+1} \neq \emptyset$.

Finally, for the possibly one i for which $W_i \neq \emptyset$ and $W_{i+1} = \emptyset$, let $W_i = \{Q_1, \ldots, Q_m\}$

with $Q_1 <_{RL} \cdots <_{RL} Q_m$ and let $G_j = 0b_1^j b_2^j b_3 b_4$ where $Q_j = b_1^j b_2^j b_3 b_4$. For each j, $G_j - b_3$ is a facet of ∂B since $W_{i+1} = \emptyset$. We can see that for $1 \leq j \leq m-1$, we have $G_j - b_3$ is adjacent to $G_{j+1} - b_3$ and $G_m - b_1^m$ is adjacent to $G_m - b_3$ in $\mathcal{H}_2$. Now for each odd $j < m$ insert $G_j - 0$ and $G_{j+1} - 0$ between $G_j - b_3$ and $G_{j+1} - b_3$ in $\mathcal{H}_2$. Note that $G_j - 0$ is adjacent to $G_{j+1} - 0$ since $G_j = L(G_{j+1}, b_2^{j+1})$. If m is odd, place $G_m - 0$ between $G_m - b_1^m$ and $G_m - b_3$. Doing all this creates a Hamiltonian cycle $\mathcal{H}$ of the facets of the squeezed sphere ∂B. $\square$

Figure 9 depicts the Hamiltonian cycle of Theorem 4.2 of the boundary facets of the 4-ball in Figure 6.

5 Remarks

As mentioned at the beginning, the fact that the set of squeezed 3-spheres contains at least one polytopal representative for each simplicial f-vector implies that the f-vector cannot be the sole obstacle for a simple 4-polytope to be Hamiltonian. It would be nice to know whether the higher dimensional squeezed spheres are Hamiltonian as well. Despite the perhaps annoyingly specialized arguments developed to tackle this particular class of objects, squeezed spheres provide a fertile ground for testing or extending properties of simplicial polytopes and exploring the boundary between polytopal and non-polytopal spheres.

	0				4	5	6			
	0				4		6	7		
$F_3 - 0$				3	4		6	7		
	0			3	4			7		
	0			3				7	8	
	0			3					8	9
	0		2	3						9
$F_7 - 0$			2	3					8	9
$F_6 - 0$		1	2						8	9
	0	1	2							9
	0	1							8	9
	0	1						7	8	
$F_5 - 0$		1	2					7	8	
$F_4 - 0$			2	3				7	8	
$F_2 - 0$			2	3			6	7		
$F_1 - 0$		1	2				6	7		
	0	1					6	7		
	0	1				5	6			
	0	1			4	5				
	0	1		3	4					
	0	1	2	3						
		1	2	3	4					
			2	3	4	5				
		1	2		4	5				
		1	2			5	6			
			2	3		5	6			
				3	4	5	6			

Figure 9: A Hamiltonian cycle of the facets of the boundary of the squeezed 4-ball

References

[1] L. J. Billera and C. W. Lee, A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes, *J. Combin. Theory Ser. A* **31** (1981) 237–255.

[2] B. Grünbaum, *Convex Polytopes*, vol. 16 of *Pure Appl. Math.*, Interscience Publishers [John Wiley & Sons, Inc.], New York, 1967.

[3] D. A. Holton and B. D. McKay, The smallest non-Hamiltonian 3-connected cubic planar graphs have 38 vertices, *J. Combin. Theory Ser. B* **45** (1988) 305–319.

[4] ——, Erratum: "The smallest non-Hamiltonian 3-connected cubic planar graphs have 38 vertices" [J. Combin. Theory Ser. B **45** (1988) 305–319; MR 90f:05090], *J. Combin. Theory Ser. B* **47** (1989) 248.

[5] G. Kalai, Many triangulated spheres, *Discrete Comput. Geom.* **3** (1988) 1–14.

[6] J. Lederberg, Hamilton circuits of convex trivalent polyhedra (up to 18 vertices) *Amer. Math. Monthly* **74** (1967) 522–527.

[7] C. W. Lee, Kalai's squeezed spheres are shellable, *Discrete Comput. Geom.* **24** (2000) 391–396.

[8] M. Rosenfeld and D. Barnette, Hamiltonian circuits in certain prisms, *Discrete Math.* **5** (1973) 389–394.

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