History of Mathematics #12

1. Before 11 pm, Sunday, April 9. Go to the Forum "Infinity" and make at least one substantive contribution by 11 pm, Sunday, April 9, and at least one substantive response to others' postings before class on Tuesday, April 11. Write about the following:

   READ CHAPTER 11 OF DUNHAM FIRST. Then write about how the various issues discussed in this chapter involving infinity and limits appear in, impact, and shape our current curriculum, down to the earliest grade levels. Reflect also on your own notions of infinity–have they changed as a result of reading this chapter?

2. Before Tuesday, April 11.

   (a) Read Dunham Chapter 11 and MTA Sketch 25. Skim Boyer Chapter 25.

   (b) Think about the following questions for discussion at the Centra session:

      i. Carefully study the mathematics in chapter 11 of Dunham for good understanding. I wish to have some substantial discussion directly on this material during the Centra session.

      ii. Starting working on the homework problems so that we can talk about them a bit in class.

3. Tuesday, April 11, 7–9 pm. Attend the Centra session to discuss the readings, forum, and comments and questions on the assigned homework due on Saturday.

4. Homework problems due Saturday, April 15, 11 pm, uploaded on the moodle site (preferred method) or submitted to the email address mathhist@ms.uky.edu.

   (a) Here is a puzzler involving infinity. There are two urns. In the first urn are balls labeled with the natural numbers, $1, 2, 3, \ldots$, one ball for each number. The second urn is initially empty. After one hour you transfer the balls labeled $1, 2, 3, \ldots, 10$ into the second urn, and then remove ball 1 from the second urn and discard it. After two hours you transfer the balls labeled $11, 12, 13, \ldots, 20$ into the second urn, and then remove ball 2 from the second urn and discard it. After three hours you transfer the balls labeled $21, 22, 23, \ldots, 30$ into the second urn, and then remove ball 3 from the second urn and discard it. This continues forever. In the end, how many balls will be in the second urn? Person A says the second urn will not be empty, because the number of balls in this urn constantly increases by 9 balls every hour. Person B says the second urn will be empty, because every ball that has been placed into it is later discarded. What is your opinion? Why?

(b) It is easy to construct an injection from the integers into the rational numbers: just use the identity function $f(x) = x$, since the set of integers is a subset of the rational numbers. But what if we want a "simple" injection from the rational numbers into the integers? Here is one approach. Consider the function f whose domain is the set of nonnegative rational numbers. To apply f to nonnegative rational number x, first express x as a fraction a/b in simplest terms. Then take $f(x) = 2^a 3^b$. Explain why this is an injection (one-to-one) into the nonnegative integers, but not a surjection (onto).

(c) This question is a follow-up to the discussion in Dunham about the denumerability of the rational numbers. Here is an interesting way to generate the rational numbers in the interval $[0, 1]$ in simplest terms without repetition, discovered in the early 1800's. Begin with the list:

$$S_1 = \frac{0}{1}, \frac{1}{1}.$$

Now create a new rational number in between by adding the numerators and adding the denominators:

$$S_2 = \frac{0}{1}, \frac{1}{2}, \frac{1}{1}.$$

Repeat this process again, inserting new fractions between each adjacent pair of old ones:

$$S_3 = \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}$$

$$S_4 = \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}$$

$$S_5 = \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}.$$

I think it is surprising to learn that continuing this process will generate every single rational number, in lowest terms, in the interval $[0, 1]$. We can use this to create a list of the rational numbers in this interval by enumerating them in the order that they are created.

i. Prove that if two nonnegative rational numbers a/b, c/d satisfy

$$\frac{a}{b} < \frac{c}{d}$$

then

$$\frac{a}{b} < \frac{a+c}{b+d}$$

and

$$\frac{a+c}{b+d} < \frac{c}{d}.$$

ii. Prove by induction that if $a/b < c/d$ are adjacent in S_k then $bc - ad = 1$.

iii. Conclude directly and immediately that if a/b and c/d are adjacent in S_k, then $\text{GCD}(a, b) = 1$ and $\text{GCD}(c, d) = 1$.

(d) We have some seen examples in this chapter of Dunham in which it is not too difficult to find a bijection between two infinite sets (e.g., the integers and the even integers). But what if the bijection is not so obvious? Sometimes it is easier to find an injection from one set into the other, and vice versa. Such injections can be used to create a bijection. Here is an example that illustrates one general method.

i. Consider the sets $X = [-1, 1]$ and $Y = (-1, 1)$. Consider the functions $f : X \to Y$ defined by $f(x) = x/2$ and $g : Y \to X$ defined by $g(y) = y$. Briefly explain why each of these functions is an injection.

ii. For any element $x \in X$ we consider its *ancestors* to be the sequence of elements, alternately in X and Y, that results from alternately applying the inverses of g and f:

$$x,\ g^{-1}(x),\ f^{-1}g^{-1}(x),\ g^{-1}f^{-1}g^{-1}(x),\ f^{-1}g^{-1}f^{-1}g^{-1}(x), \ldots$$

This sequence may end (in fact, it may end with x itself having only itself as an ancestor). We can divide (partition) the elements of X into three subsets, $X = A \cup B \cup C$. A is the subset of those x for which there is a final ancestor and this ancestor is in X. B is the subset of those x those for which there is a final ancestor and this ancestor is in Y. C is the subset of those x having an infinite number of ancestors.

For the sets X and Y, and functions f and g, given in part (4(d)i), explicitly describe the sets A, B, and C.

iii. Now we construct a bijection $h : X \to Y$ as follows:

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \cup C \\ g^{-1}(x) & \text{if } x \in B \end{cases}$$

For the sets X and Y, and functions f and g, given in part (4(d)i), explicitly describe the function h, and explain why it is a bijection.