History of Mathematics #6

1. Before 11 pm, Sunday, February 19. Go to the Forum "Complex Numbers" and make at least one substantive contribution by 11 pm, Sunday, February 19, and at least one substantive response to others' postings before class on Tuesday, February 21. Write about the following:

   (a) How can complex numbers be introduced within the K–16 curriculum without the unnecessary stigma of being "imaginary"? For example, would it be more effective to introduce them at the outset as ordered pairs in the plane with particular rules for addition and multiplication that are graphically defined? (You may need first to look up the graphical definitions of the operations.)

2. Before Tuesday, February 21.

   (a) Read Dunham Chapter 6 and MTA Sketches 11 and 17. Skim Boyer Chapter 15.

   (b) Think about the following questions for discussion at the Centra session:

      i. How do you know that every cubic equation in real coefficients has at least one real solution? (Think about the behavior of $p(x)$ as x goes to $-\infty$ and to $+\infty$.)

      ii. How does Dunham make the point that it was the quest for the solution of the cubic equation that forced acceptance of complex numbers, rather than the quadratic formula?

      iii. Study carefully the derivation of the solution of the depressed cubic.

      iv. How does one prove that that $x - r$ is a factor of a polynomial $p(x)$ if and only if $p(r) = 0$? How does this help if one is trying to find all of the roots of a cubic or higher order polynomial?

      v. Explain why if $p(x)$ is a polynomial with real coefficients, and $a + bi$ is a root, then so is $a - bi$. (Think about properties of complex conjugates, and what happens to conjugates of sums and products.) Then explain how this shows that any polynomial with real coefficients can be factored into linear and quadratic polynomials with real coefficients.

      vi. How can one reduce a general cubic to a depressed cubic?

      vii. Explain this statement of Boyer on page 288: "Any attempt to find algebraically the cube roots of the imaginary numbers in the Cardan-Tartaglia rule leads to the very cubic in the solution of which the cube roots arose in the first place, so that one is back where he started from. Becuase this impasse arises whenever all three roots are real, this is known as the 'irreducible case.' Here an expression for the unknown is indeed provided by the formula, but the form in which this appears is useless for most purposes."

viii. See what happens when you use Maxima (http://maxima.sourceforge.net/index.shtml) to solve a cubic. (Or Maple, Mathematica, or suitable calculator with built-in computer algebra system.)

ix. How can one obtain approximations to the roots of a general polynomial? Look up Newton's method.

x. What does it mean to say that there is no formula for polynomial equations of order five and above? Why doesn't this contradict the statement that every polynomial with real or complex coefficients can be factored into linear polynomials with complex coefficients?

xi. Here are three ways to "invent" the complex numbers. I saw all three in high school and/or college. I think our high school teachers should know them. Think about each of these cases. In each case: Why are the operations commutative, associative, and distributive? What are the additive and multiplicative identities? What is i? How do the real numbers appear as a subset (subfield)? How can you see that $i^2 = 1$?

   A. A complex number is an ordered pair (a, b) of real numbers. Addition is defined by $(a, b) + (c, d) = (a + c, b + d)$. Multiplication is defined by $(a, b) * (c, d) = (ac - bd, ad + bc)$. (I definitely saw this in high school.)

   B. A complex number is a point (a, b) in the xy-plane, associated with a vector pointing from $(0, 0)$ to (a, b). Each vector has a length $\sqrt{a^2 + b^2}$ and an angle that the vector makes with respect to the positive x-axis (think about conversions to and from polar coordinates). Addition is defined by the standard parallelogram law for vectors. To multiply two complex numbers, add their angles and multiply their lengths. (I definitely saw this in high school.)

   C. A complex number is a 2×2 matrix of real numbers of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

   Addition and multiplication is defined in the usual way as for matrices. (I am pretty sure I saw this in high school, but it may have been in college.)

3. Tuesday, February 21, 7–9 pm. Attend the Centra session to discuss the readings, forum, and comments and questions on the assigned homework due on Friday.

4. Homework problems due Friday, February 24, 11 pm, uploaded on the moodle site (preferred method) or submitted to the email address mathhist@ms.uky.edu.

   (a) (From MTA, Expanded Edition, page 137): What happens if you try to apply Cardano's formula to the following equations?

      i. $x^3 + 9x - 26 = 0$
      ii. $x^3 + 3x - 4 = 0$
      iii. $x^3 - 7x + 6 = 0$

      Do not use outside sources for this problem, though you may talk with each other.

   (b) (From MTA, Expanded Edition, page 137): Sketch 11 says that Bombelli showed that $x^3 = px + q$ always has a positive solution, regardless of the (positive) values of p and q. Give a graphical explanation of why this claim is plausible.

      Do not use outside sources for this problem, though you may talk with each other.

   (c) (From MTA, Expanded Edition, Sketch 17, page 183): Check that multiplying $\cos x + i \sin x$ by $\cos y + i \sin y$ results in $\cos(x + y) + i \sin(x + y)$. Explain why De Moivre's formula

$$(\cos(x) + i \sin(x))^n = \cos(nx) + i \sin(nx)$$

      follows.

      Do not use outside sources for this problem, though you may talk with each other.

   (d) (From MTA, Expanded Edition, pages 183–184; refer to Sketch 17): Argand's planar representation of the complex numbers leads to useful visualizations of complex arithmetic. If you think of each point of the plane as the tip of an arrow (a vector) with its tail at $(0, 0)$, then that point can be specified by the length of the arrow and the angle it makes with the positive x-axis.

      i. Calculate the vector lengths and angles for $3+4i$, $-5+7i$, and $1-i$. Describe how to do this for any complex number $a + bi$.
      ii. Which complex number has a vector length of 6 and an angle of 45 degrees ($\pi/4$ radians)? Which has a length of 1 and an angle of $\pi/2$ radians?
      iii. Vectors can be used to represent the sum of two complex numbers as the "parallelogram sum" of vector addition. Illustrate this by drawing diagrams for $(2 + 5i) + (4 + 3i)$ and $(3 + 6i) + (5 - 2i)$.
      iv. Multiplication of numbers of the unit circle (the dotted circle in Display 2 of the Sketch) is particularly nice. Using the facts given in the Sketch, prove that the product of two complex numbers on this circle can be found by adding their angles.

v. Display 2 and part (4(d)iv) of this question should make it clear that there are four distinct 4th roots of 1. Generalize this to explain why there are n distinct nth roots of 1 for each natural number n. Write out the six 6th roots of 1.

Do not use outside sources for this problem, though you may talk with each other.

(e) We can use power series to define $\sin x$, $\cos x$ and e^x for real and complex numbers x:

$$\sin x = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$\cos x = \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

(It's fun to use a calculator to do some computations using these formulas to see how rapidly they converge!) Replace x with iy in the formula for e^x to motivate Euler's famous formula:

$$e^{iy} = \cos y + i \sin y.$$

In particular, verify that $e^{\pi i}+1 = 0$, a marvelous equation involving five important mathematical constants.

Do not use outside sources for this problem, though you may talk with each other.