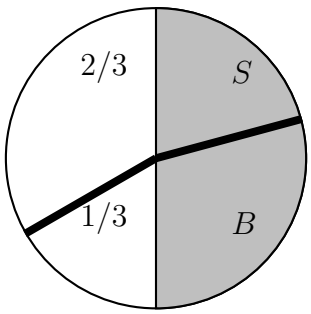
MA 111 Supplemental Notes on Divider-Chooser

We discussed the basic idea of the Divider-Chooser (a.k.a, You cut-I choose) method of fair division last Wednesday. While the underlying idea is fairly straightforward, the actual division process (and the valuation process) can be rather involved.

Sally and Olivia are going to share a pizza. The pizza is half pepperoni and half olives. (The olive half is the shaded half). Sally likes pepperoni and olives equally. Olivia likes pepperoni and olives, but she likes olives twice as much as she likes sausage.

Suppose Olivia is the divider, and she divides the pizza in the following way:
The pepperoni half is divided into two pieces, one twice as large as the other
The olive half is divided into two pieces so that the small pepperoni piece plus the large olive piece has the same value (according to Olivia) as the large pepperoni piece plus the small olive piece.

Qualitatively, the division will look like:



The 2/3 and 1/3 represent the proportion of the pepperoni half included in each piece. The S and the B represent the proportion of the olive half included in each piece.

Now, as far as Olivia is concerned, the pepperoni half only contains 1/3 the value of the pizza whereas the olive half contains 2/3 the value of the pizza. Thus, Olivia assigns the values $(1/3)\cdot(1/3) = 1/9$ and $(1/3)\cdot(2/3) = 2/9$ to the two pepperoni slices. Likewise, assigns the values $(2/3)\cdot S$ and $(2/3)\cdot B$ to the two olive slices.

The small pepperoni slice together with the large olive slice is worth

$$\frac{1}{9} + \frac{2}{3}\cdot B = \frac{1}{2}$$

So

$$\frac{2}{3}B = \frac{1}{2} - \frac{1}{9} = \frac{9}{18} - \frac{2}{18} = \frac{7}{18}$$

So

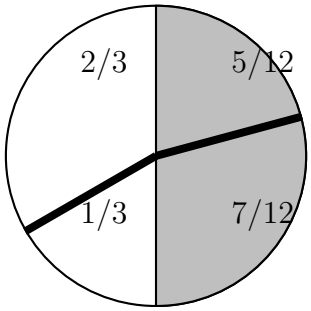
$$\frac{2}{3}B = \frac{7}{18} \Longrightarrow B = \frac{7}{18}\cdot\frac{3}{2} = \frac{7}{12}$$

Therefore, the large olive slice should consist of $\frac{7}{12}$ of the olive half. The smaller olive slice coniststs of

$$1 - \frac{7}{12} = \frac{12}{12} - \frac{7}{12} = \frac{5}{12}$$

of the olive half.

Filling in the values of S and B, the division is



How does Sally value the two pieces? Sally values pepperoni and olive equally: the pepperoni half is worth half the pizza; the olive half is worth half the pizza. Therefore, the small pepperoni slice together with the large olive slice is worth

$$\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{7}{12} = \frac{1}{6} + \frac{7}{24} = \frac{4}{24} + \frac{7}{24} = \frac{11}{24}$$

Likewise, the large pepperoni slice together with the small olive slice is worth

$$\frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{5}{12} = \frac{2}{6} + \frac{5}{24} = \frac{8}{24} + \frac{5}{24} = \frac{13}{24}$$

Thus, as far as Sally is concerned, the large pepperoni slice together with the small olive slice is the better piece. ($13/24 \approx 54.2\%$ compared to $11/24 \approx 45.8\%$)

Suppose Sally recently had a bad experience with pepperoni, and now she likes olives 3 times more than she likes pepperoni. Given Sally's new prefences, how will she value the two pieces?

Since Sally likes olives 3 times more than she likes pepperoni, she assigns a value of $\frac{1}{4}$ to the pepperoni half and a value of $\frac{3}{4}$ to the olive half.

The small pepperoni slice together with the large olive slice is now worth

$$\frac{1}{4} \cdot \frac{1}{3} + \frac{3}{4} \cdot \frac{7}{12} = \frac{1}{12} + \frac{21}{48} = \frac{4}{48} + \frac{21}{48} = \frac{25}{48}$$

The large pepperoni slice together with the small olive slice is now worth

$$\frac{1}{4} \cdot \frac{2}{3} + \frac{3}{4} \cdot \frac{5}{12} = \frac{2}{12} + \frac{15}{48} = \frac{8}{48} + \frac{15}{48} = \frac{23}{48}$$

Given Sally's new value system, the small pepperoni slice together with the large olive slice is the better piece. ($25/48 \approx 52.1\%$ compared to $23/48 \approx 47.9\%$)