

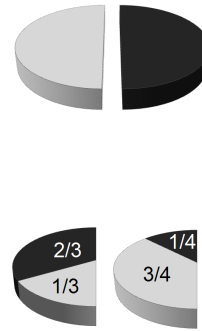
### 11.18.1 Introduction to Fair-Division Games — Adaption of Notes by Elizabeth Weaver

- In this chapter, we will discuss several ways that something can be divided among competing parties in a way that ensures that each party receives what they consider to be a fair share.
- We will think of fair division in terms of games — with players, goals, and rules.
- **Basic Elements of Fair-Division Games**
  - The set of goods to be divided, denoted by  $S$ .
  - The players who have a right to the goods, denoted by  $P_1, P_2, \dots, P_N$ .
  - The value systems that give each player the ability to quantify the value of the goods.
- **Basic Assumptions**
  - Rationality — Each of the players acts rationally.
  - Cooperation — The players are willing participants and accept the rules of the game as binding.
  - Privacy — The players know nothing of the other players' value systems.
  - Symmetry — Every player is entitled to an equal share of the goods.
- **Fair Division:** A fair division of  $S$  is a division in which every player gets a fair share.
- **Fair Share:** Suppose that  $s$  is a share of the goods to be divided,  $S$ , and that  $P$  is one of the players in a fair-division game with  $N$  players. We say that  $s$  is a fair share to player  $P$  if  $s$  is worth *at least*  $1/N$ th of the total value of  $S$  *in the opinion of*  $P$ .
- **Example:** Three players (DJ, Stephanie, and Michelle) are sharing a cake. Suppose that the cake is divided into three slices ( $s_1$ ,  $s_2$ , and  $s_3$ ). The following table gives the value of each slice in the eyes of the players.

	$s_1$	$s_2$	$s_3$
DJ	\$3.00	\$5.00	\$4.00
Stephanie	\$4.00	\$4.50	\$6.50
Michelle	\$4.50	\$4.50	\$4.50

- (a) Which of the three slices are fair shares to DJ?
  - (b) Which of the three slices are fair shares to Stephanie?
  - (c) Which of the three slices are fair shares to Michelle?
  - (d) Find a fair division of the cake using  $s_1$ ,  $s_2$ , and  $s_3$ .
  - (e) Explain why there is only one such fair division possible.
- There are three different types of fair-division games.
    - A fair division game is **continuous** if the set  $S$  is divisible in infinitely many ways. Typical examples involve the division of land, pizza, cake, etc.
    - A fair division game is **discrete** if the set  $S$  is made of objects that are not divisible like houses, jewelry, art, etc.
    - A fair division game is **mixed** if the set  $S$  is one in which  $S$  is made up of components that are both continuous and discrete. In these types of games, the continuous and discrete components can be divided up separately.
  - **Example:** Jerry and George jointly buy the half chocolate-half vanilla cake for \$24. Suppose that Jerry feels that 25% of the value of the cake lies in the vanilla half and 75% of the value lies in the chocolate half. Find the dollar value to Jerry for each of the following pieces.

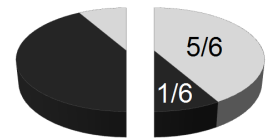
- (a) The vanilla half of the cake.
- (b) The chocolate half of the cake.
- (c) A slice which is  $\frac{1}{3}$  of the vanilla and  $\frac{2}{3}$  of the chocolate.
- (d) A slice which is  $\frac{1}{4}$  of the chocolate and  $\frac{3}{4}$  of the vanilla.
- (e) How would Jerry slice the cake into two fair shares?



### 11.18.2 The Divider-Chooser Method

- This method can be used any time the fair-division game involves two players and a continuous set  $S$ .
- **Divider-Chooser Method**
  1. One player, called the divider, divides  $S$  into two shares that he considers to be fair.
  2. The second player, called the chooser, picks the share that he wants, leaving the other share to the divider.
- Why is the divider guaranteed a fair share in this method? Why is the chooser guaranteed a fair share?
- Why will the divider always split  $S$  in a way he determines to be 50/50?

- **Example:** Diane and Carla want to split a cheesecake that is half chocolate and half vanilla. Diane likes chocolate and vanilla equally well. To her, both halves of the cake, shown below, are equal in value. Carla, however is allergic to chocolate, so for her the chocolate half has 0% of the value and the vanilla half has 100% of the value. Suppose Diane cuts the cake into the two pieces shown below. Which piece will Carla choose? Why is this arrangement fair? Who gets the better deal?
- Is it better to be the divider or the chooser?



### 11.18.3 After this, you should be able to...

- understand the concept of a fair-division game.
- use the notations and ideas introduced in the basic elements of fair-division games.
- state the basic assumptions of our fair-division games.
- determine whether a fair-division game is continuous, discrete, or mixed.
- determine a share's worth to a player given a certain set up.
- determine whether shares are fair shares to certain players given a certain set-up.
- state the steps in the divider-chooser method.
- determine the outcome of the divider-chooser method given a certain set-up.
- solve problems in parts A and B of pages 103–107. Begin with problems #1, 3, 5, 7, 11, 15, 17, 19.