Shapes and Designs
Extensions 1

1. Four squares can fit together perfectly in the plane surrounding a common vertex (since each interior angle of a square measures 90 degrees). Let's call this a (4,4,4,4) cluster. Similarly, two squares and three equilateral triangles can fit together perfectly surrounding a common vertex. There are essentially two different ways to do this: (4,4,3,3,3) (where the squares are adjacent) and (4,3,4,3,3) (where the squares are not adjacent), and we will regard these as two *different* clusters. Note that we could have called this last cluster $(3, 3, 4, 3, 4)$ as well—it still refers to the same cluster. However, $(4, 4, 3, 3, 3)$ and $(4, 3, 4, 3, 3)$ are *not* the same.

   (a) You have just seen three *planar clusters*. Determine all possible planar clusters that can be formed by fitting together combinations of regular polygons in the plane surrounding a common vertex. Be *systematic* in some fashion, so that you can be certain you have found all of them, and explain clearly how you know this.

   (b) Some of the clusters can be extended to tile the plane so that at every vertex, exactly the same cluster appears—the same sequence of polygons, in either clockwise or counterclockwise order. For example, if you extend the $(4, 4, 4, 4)$ cluster, you get the familiar tiling of the plane with squares, with four squares meeting at each vertex. Of the clusters you have found, determine which ones can be extended. Make a precise drawing of each one you have found. You may wish to use Geometer's Sketchpad to make the drawings.

   (c) Choose one of the clusters that cannot be extended to tile the plane and prove that it cannot.

2. If the measures of the interior angles of a cluster of regular polygons surrounding a common vertex sum to less than 360 degrees, then the cluster will not be planar. Let's call such clusters *space clusters*. For example, the $(4, 4, 4)$ cluster, consisting of three squares meeting at a common vertex, is a space cluster. If you try to extend this cluster so that the same cluster surrounds each vertex, you will construct a cube.

   (a) Identify all possible space clusters consisting of a *single* type of regular polygon, and prove that you have found them all. For each one, extend the cluster to create a three-dimensional polyhedron. Build the polyhedron using Polydron. Look up the name of each polyhedron.

   (b) Show that the space cluster $(3, 4, 3, 4)$ can be extended to make a polyhedron. Build the polyhedron using Polydron. Draw a careful picture of it. Look up the name of this polyhedron.

(c) Prove that the space cluster $(3, 3, 4, 4)$ cannot be extended to make a polyhedron.

(d) What familiar object is based on the space cluster $(6, 6, 5)$?

3. Spherical tilings You may find the Lénárt sphere helpful when thinking about these problems.

   (a) What is the analog of a line segment on the surface of a sphere? Why do you think so?

   (b) What is the analog of a triangle on the surface of a sphere? Why do you think so?

   (c) What is the analog of an equilateral triangle on the surface of a sphere? Why do you think so?

   (d) Prove that you can tile the sphere with eight equilateral triangles on the sphere. What is true about the angles of each of these triangles? In what ways are they like planar equilateral triangles? In what ways are they different?

4. Tilings with irregular polygons

   (a) Look at ACE question 3. Prove or disprove: Every triangle can be used to tile the plane. Include good diagrams.

   (b) Look at ACE question 13. Prove or disprove: Every quadrilateral can be used to tile the plane. Include good diagrams. Don't forget to consider the possiblity that the quadrilateral is not convex.

   (c) Find a copy of a tiling by M.C. Escher that uses irregular shapes and make a good photocopy or printout of it on a separate sheet of paper. Include a clear reference to where you found it on the back. In what ways is this tiling symmetrical?

5. Honeycombs

   (a) Find a description of the polyhedron called the *rhombic dodecahedron* and an explanation of what this polyhedron has to do with honeycombs. Write a summary in your own words, but cite your sources properly.

   (b) Construct a model of the rhombic dodecahedron with Zomesystem using the yellow struts.