Representation

Instructional programs from prekindergarten through grade 12 should enable all students to—

- create and use representations to organize, record, and communicate mathematical ideas;
- select, apply, and translate among mathematical representations to solve problems;
- use representations to model and interpret physical, social, and mathematical phenomena.

The ways in which mathematical ideas are represented is fundamental to how people can understand and use those ideas. Consider how much more difficult multiplication is using Roman numerals (for those who have not worked extensively with them) than using Arabic base-ten notation. Many of the representations we now take for granted—such as numbers expressed in base-ten or binary form, fractions, algebraic expressions and equations, graphs, and spreadsheet displays—are the result of a process of cultural refinement that took place over many years. When students gain access to mathematical representations and the ideas they represent, they have a set of tools that significantly expand their capacity to think mathematically.

The term *representation* refers both to process and to product—in other words, to the act of capturing a mathematical concept or relationship in some form and to the form itself. The child who wrote her age as shown in figure 3.8 used a representation. The graph of $f(x) = x^3$ is a representation. Moreover, the term applies to processes and products that are observable externally as well as to those that occur "internally," in the minds of people doing mathematics. All these meanings of representation are important to consider in school mathematics.

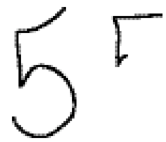

Fig. **3.8.** A child's representation of five and one-half

Some forms of representation—such as diagrams, graphical displays, and symbolic expressions—have long been part of school mathematics. Unfortunately, these representations and others have often been taught and learned as if they were ends in themselves. Representations should be treated as essential elements in supporting students' understanding of mathematical concepts and relationships; in communicating mathematical approaches, arguments, and understandings to one's self and to others; in recognizing connections among related mathematical concepts; and in applying mathematics to realistic problem situations through modeling. New forms of representation associated with electronic technology create a need for even greater instructional attention to representation.

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Create and use representations to organize, record, and communicate mathematical ideas

Students should understand that written representations of mathematical ideas are an essential part of learning and doing mathematics. It is important to encourage students to represent their ideas in ways that make sense to them, even if their first representations are not conventional ones. It is also important that they learn conventional forms of representation to facilitate both their learning of mathematics and their communication with others about mathematical ideas.

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The fact that representations are such effective tools may obscure how difficult it was to develop them and, more important, how much work it takes to understand them. For instance, base-ten notation is difficult for young children, and the curriculum should allow many opportunities for making connections between students' emerging understanding of the counting numbers and the structure of base-ten representation. But as students move through the curriculum, the focus tends to be increasingly on presenting the mathematics itself, perhaps » under the assumption that students who are old enough to think in formal terms do not, like their younger counterparts, need to negotiate between their naive conceptions and the mathematical formalisms. Research indicates, however, that students at all levels need to work at developing their understandings of the complex ideas captured in conventional representations. A representation as seemingly clear as the variable *x* can be difficult for students to comprehend.

The idiosyncratic representations constructed by students as they solve problems and investigate mathematical ideas can play an important role in helping students understand and solve problems and providing meaningful ways to record a solution method and to describe the method to others. Teachers can gain valuable insights into students' ways of interpreting and thinking about mathematics by looking at their representations. They can build bridges from students' personal representations to more-conventional ones, when appropriate. It is important that students have opportunities not only to learn conventional forms of representation but also to construct, refine, and use their own representations as tools to support learning and doing mathematics.

Through the middle grades, children's mathematical representations usually are about objects and actions from their direct experience. Primary school students might use objects to represent the number of wheels on four bicycles or the number of fireflies in a story. They may represent larger numbers of objects using place-value mats or base-ten blocks. In the middle grades, students can begin to create and use mathematical representations for more-abstract objects, such as rational numbers, rates, or linear relationships. High school students should use conventional representations as a primary means for expressing and understanding more-abstract mathematical concepts. Through their representations, they should be ready to see a common structure in mathematical phenomena that come from very different contexts.

Representations can help students organize their thinking. Students' use of representations can help make mathematical ideas more concrete and available for reflection. In the lower grades, for example, children can use representations to provide a record for their teachers and their peers of their efforts to understand mathematics. In the middle grades, they should use representations more to solve problems or to portray, clarify, or extend a mathematical idea. They might, for example, focus on collecting a large amount of data in a weather experiment over an extended time and use a spreadsheet and related graphs to organize and represent the data. They might develop an algebraic representation for a real-world relationship (e.g., the number of unit tiles around a rectangular pool with dimensions m units by n units where m and n are integers; see fig. 3.9) and begin to recognize that symbolic representations that appear different can describe the same phenomenon. For instance, the number of tiles in the border of the $m \times n$ pool can be expressed as $2n + 2m + 4$ or as $2(m + 2) + 2n$.

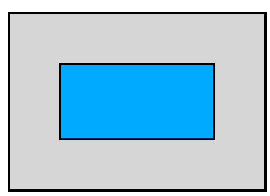

Fig. **3.9.** Rectangular pool and border

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Computers and calculators change what students can do with conventional representations and expand the set of representations with which they can work. For example, students can flip, invert, stretch, and zoom in on graphs using graphing utilities or dynamic geometry software. They can use computer algebra systems to manipulate expressions, and they can investigate complex data sets using spreadsheets. As students learn to use these new, versatile tools, they also can consider ways in » which some representations used in electronic technology differ from conventional representations. For example, numbers in scientific notation are represented differently in calculators and in textbooks. The algebraic expressions on computer algebra systems may look different from those in textbooks.

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Select, apply, and translate among mathematical representations to solve problems

Different representations often illuminate different aspects of a complex concept or relationship. For example, students usually learn to represent fractions as sectors of a circle or as pieces of a rectangle or some other figure. Sometimes they use physical displays of pattern blocks or fraction bars that convey the part-whole interpretation of fractions. Such displays can help students see fraction equivalence and the meaning of the addition of fractions, especially when the fractions have the same denominator and when their sum is less than 1. Yet this form of representation does not convey other interpretations of fraction, such as ratio, indicated division, or fraction as number. Other common representations for fractions, such as points on a number line or ratios of discrete elements in a set, convey some but not all aspects of the complex fraction concept. Thus, in order to become deeply knowledgeable about fractions—and many other concepts in school mathematics—students will need a variety of representations that support their understanding.

The importance of using multiple representations should be emphasized throughout students' mathematical education. For example, a prekindergarten through grade 2 student should know how to represent three groups of four through repeated addition, skip-counting, or an array of objects. Primary-grades students begin to see how some representations make it easier to understand some properties. Using the arrays in figure 3.10, a teacher could make commutativity visible.

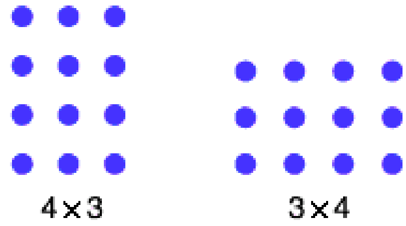

Fig. **3.10.** These arrays can help teachers explain commutativity.

In grades 3–5, students' repertoires of representations should expand to include more-complex pictures, tables, graphs, and words to model problems and situations. For middle-grades students, representations are useful in developing ideas about algebra. As students become mathematically sophisticated, they develop an increasingly large repertoire of mathematical representations as well as a knowledge of how to use them productively. Such knowledge includes choosing and moving between representations and learning to ask such questions as, Would a graph give me more insight than a symbolic expression to solve this problem?

One of the powerful aspects of mathematics is its use of *abstraction*—the stripping away by symbolization of some features of a problem that are not necessary for analysis, allowing the "naked symbols" to be operated on easily. In many ways, this fact lies behind the power of mathematical applications and modeling. Consider this problem:

> From a ship on the sea at night, the captain can see three lighthouses and can measure the angles between them. If the captain knows the position of the lighthouses from a map, can the captain determine the position of the ship?

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When the problem is translated into a mathematical representation, the ship and the lighthouses become points in the plane, and the problem » can be solved without knowing that it is about ships. Many other problems from different contexts may have similar representations. As soon as the problem is represented in some form, the classic solution methods for that mathematical form may be used to solve the problem.

Technological tools now offer opportunities for students to have more and different experiences with the use of multiple representations. For instance, in prekindergarten through grade 2, teachers and students can work with on-screen versions of concrete manipulatives, getting accuracy and immediate feedback. Later on, dynamic geometry tools can be used for generating conjectures. Several software packages allow students to view a function simultaneously in tabular, graphical, and equation form. Such software can allow students to examine how certain changes in one representation, such as varying a parameter in the equation $ax^2 + bx + c = 0$, simultaneously affect the other representations. Computer and calculator simulations can be used to investigate physical phenomena, such as motion.

As students' representational repertoire expands, it is important for students to reflect on their use of representations to develop an understanding of the relative strengths and weaknesses of various representations for different purposes. For example, when students learn different representational forms for displaying statistical data, they need opportunities to consider the kinds of data and questions for which a circle graph might be more appropriate than a line graph or a box-and-whiskers plot more appropriate than a histogram.

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Use representations to model and interpret physical, social, and mathematical phenomena

The term *model* has many different meanings. So it is not surprising that the word is used in many different ways in discussions about mathematics education. For example, *model* is used to refer to physical materials with which students work in school—manipulative models. The term is also used to suggest exemplification or simulation, as in when a teacher models the problem-solving process for her students. Yet another usage treats the term as if it were roughly synonymous with *representation.* The term *mathematical model,* which is the focus in this context, means a mathematical representation of the elements and relationships in an idealized version of a complex phenomenon. Mathematical models can be used to clarify and interpret the phenomenon and to solve problems.

In some activities, models allow a view of a real-world phenomenon, such as the flow of traffic, through an analytic structure imposed on it. An example of a general question to be explored might be, How long should a traffic light stay green to let a reasonable number of cars flow through the intersection? Students can gather data about how long (on average) it takes the first car to go through, the second car, and so on. They can represent these data statistically, or they can construct analytic functions to work on the problem in the abstract, considering the wait time before a car starts moving, how long it takes a car to get up to regular traffic speed, and so on.

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Technological tools now allow students to explore iterative models for situations that were once studied in much more advanced courses. For example, it is now possible for students in grades 9–12 to model » predator-prey relations. The initial set-up might be that a particular habitat houses so many wolves and so many rabbits, which are the wolves' primary food source. When the wolves are well fed, they reproduce well (and more wolves eat more rabbits); when the wolves are starved, they die off. The rabbits multiply readily when wolves are scarce but lose numbers rapidly when the wolf population is large. Modeling software that uses difference equations allows students to enter initial conditions and the rules for change and then see what happens to the system dynamically.

Students' use of representations to model physical, social, and mathematical phenomena should grow through the years. In prekindergarten through grade 2, students can model distributing 24 cookies among 8 children, using tiles or blocks in a variety of ways. As students continue to encounter representations in grades 3–5, they begin to use them to model phenomena in the world around them and to aid them in noticing quantitative patterns. As middle-grades students model and solve problems that arise in the real and mathematical worlds, they learn to use variables to represent unknowns and also learn to employ equations, tables, and graphs to represent and analyze relationships. High school students create and interpret models of phenomena drawn from a wider range of contexts—including physical and social environments—by identifying essential elements of the context and by devising representations that capture mathematical relationships among those elements. With electronic technologies, students can use representations for problems and methods that until recently were difficult to explore meaningfully in high school. Iterative numerical methods, for example, can be used to develop an intuitive concept of limit and its applications. The asymptotic behavior of functions is more easily understood graphically, as are the effects of transformations on functions. These tools and understandings give students access to models that can be used to analyze a greatly expanded range of realistic and interesting situations.