Notes

Contents

1 Polya's Four Phases of Problem Solving

The following comes from the famous book by George Polya called *How to Solve it.*

1. **Understanding the Problem.**

   You have to *understand* the problem. *What is the unknown? What are the data? What is the condition?* Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?

   Draw a figure. Introduce suitable notation.

   Separate the various parts of the condition. Can you write them down?

2. **Devising a Plan.**

   Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.

   Have you seen it before? Or have you seen the same problem in a slightly different form?

   Do you know a related problem? Do you know a theorem that could be useful?

   Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.

   Here is a problem related to yours and solved before. Could you use it? Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?

   Could you restate the problem? Could you restate it still differently? Go back to definitions.

   If you cannot solve the proposed problem try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or the data, or both if necessary, so that the new unknown and the new data are nearer to each other?

   Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

3. **Carrying Out the Plan.**

   Carry out your plan.

   Carrying out your plan of the solution, *check each step.* Can you see clearly that the step is correct? Can you prove that it is correct?

4. **Looking Back.**

   Examine the solution obtained.

   Can you *check the result?* Can you check the argument?

   Can you derive the result differently? Can you see it at a glance?

   Can you use the result, or the method, for some other problem?

2 Some of my Informal Thoughts on Teaching

1. Don't neglect history. Mathematics is far, far from static. Everything you have studied was invented, discovered, or developed at some point in the past, and sometimes not so long ago! Many have the very strange impression that most of mathematics was finished centuries ago and is now fixed and unchanging. For many, the idea of mathematical research is hard to comprehend, but this should not be so. Mathematics is dynamic, alive, and growing by leaps and bounds!

2. Look for concrete (especially physical) examples, models, or visualizations of mathematical concepts. Conversely, look for the mathematics in everyday objects. For example, what can you say about the lengths of the chimes hanging beneath a xylophone?

3. Be conscious of the difference between skills and concepts. Skills are sometimes more easily learned, practiced, and tested than concepts—hence very tempting to both students and teachers to spend most of their time on—but a math course that degenerates into a sequence of rote skills and manipulation without any real understanding is essentially worthless.

4. Practice asking lots of questions. A good question is worth a lot more than a mediocre fact.

5. Think about analogies with learning in other disciplines—for example, learning to read, learning how to play a musical instrument. In both cases there is an absolute necessity for constant practice, without which it is impossible to attain any level of sophistication or appreciation. This can be hard work! You cannot read Shakespeare if you are still sounding out individual words letter by letter. Also, under no circumstances would I want my first grade child to be taught to read by a teacher who could only read on an elementary school level, or my twelfth grade child to take an English class with a teacher who could only function on a twelfth grade level. The great teachers have a sense of perspective on their subjects which transcends the level on which they are teaching, and students are acutely aware of this.

6. There is a high probability that you will frequently encounter students that are more adept mathematically than you. You must be prepared for this, or you may tragically close doors of opportunity forever for your students.

7. Your love of mathematics should and will undoubtedly extend to activities outside of the classroom. Great musicians love music and play music, even when they are not

performing. Great writers do not confine their creativity to an 8-hour day.

8. You must learn how to learn new mathematics on your own, and teach your students this crucial skill.

9. Mathematics is not a linear subject, but parts are intricately intertwined into a complex structure. In school and college, certain strands are extracted and taught, sometimes giving a very misleading view of mathematics as a whole. Think about ways to reweave the mathematical fabric as you teach.

10. Some of the topics that are studied in a semester took decades, centuries, or even millennia to come to fruition. Is it little wonder that it is hard for students to fully comprehend some of the things we are teaching them the first time (or even the second or third time) that they see them? A good example of this are the concepts of continuity and differentiation in calculus. I recommend that you read about the struggle to try to place the less formal or intuitive views of these concepts on a sound footing. How long did it take?

11. Mathematics is an experimental subject, but the truths discovered are subject to verification in a rigorous way. All too often, the final results are presented in such a way that is cleansed of the explanation of the process of investigation and discovery. This is very sad, for much of value has been lost. One good example of this principle is Archimedes' "Method."

12. High quality work is often hard work. Don't cheat yourself (or others) out of the benefits of significant accomplishment.

There are thousands of books and articles that capture the spirit of mathematics and provide an unimaginable amount of wonderful material. I mention a few of my favorites. There are probably newer editions of all of these.

1. All of the books by Martin Gardner on recreational mathematics and related subjects. Most of this material was first published in the Mathematical Recreations column of *Scientific American*. I grew up on this material starting in elementary school.

2. H. Steinhaus, *Mathematical Snapshots*, Oxford University Press. A gallery of fascinating mathematical snippets.

3. G. Polya, *How to Solve it*, Princeton University Press. My high school teacher showed us a movie by Polya on problem solving that made a deep and long-lasting impression on me.

4. W.W. Rouse Ball and H.S.M. Coxeter, *Mathematical Recreations and Essays*, University of Toronto Press, 1974. I read and reread this book as a high school student.

3 Suggestions for Approaching Mathematics in your Courses

Definitions. Construct examples. Make diagrams. Construct non-examples by relaxing various conditions.

Theorems. Restate them. Make simple examples. Make complicated examples. Make diagrams. Test the validity of the theorem when certain of the hypotheses are relaxed. Consider the validity of the converse. Generalize the theorem. Verbalize connections with other mathematics in the course—how does it fit in? Consider further mathematical questions that are suggested by the existence of this theorem.

Proofs. Make outlines. Make diagrams. Construct a "sufficiently complicated example" with which to follow along the proof.

The Course. Verbalize how the course fits together as a whole. What are the major themes? How to the topics connect and flow together? Consider an outlining strategy: Make detailed marginal notes from which you can reconstruct the mathematics of the notes. Then make a coarser outline from which you can reconstruct the marginal notes. Then make a brief outline from which you can reconstruct the coarser outline.

Study. Read the suggestions from Joy Williams and Frank Branner in `http://www.ms.uky.edu/~math/Grad/handbook-archive/handbook-06.pdf`.

4 Review of Problem Solutions

Here are some ideas for evaluating a proposed formal (e.g., written) solution to a problem.

1. Is a statement of the problem included?
2. Is the answer correct?
3. Is the method of solution valid?
4. Is the explanation clear and complete, including sufficient verbal explanation to clarify the various steps and reasoning?
5. Is the mathematics correct?
6. Is the mathematical vocabulary correct?
7. When helpful, does the explanation make good use of such aids as diagrams, charts, tables, etc.?
8. Is the answer doublechecked at the end?
9. Does the explanation use proper grammar and spelling?

Suggestions: Write your solution so that if you read it a year it a year from now you will understand both the statement of the problem and your solution. Try reading the solution out loud to yourself, or ask a friend to do this and offer feedback using the above criteria.

5 Guessing Formulas

> It should be remarked that although the principle of mathematical induction suffices to *prove* the formula... once this formula has been written down, the proof gives no indication of how this formula was arrived at in the first place; why precisely the expression $[n(n+1)/2]^2$ should be guessed as an expression for the sum of the first n cubes, rather than $[n(n+1)/3]^2$ or $(19n^2-41n+24)/2$ or any of the infinitely many expressions of a similar type that could have been considered. The fact that the proof of a theorem consists in the application of certain simple rules of logic does not dispose of the creative element in mathematics, which lies in the choice of the possibilities to be examined. The question of the origin of the *hypothesis*... belongs to a domain in which no very general rules can be given; experiment, analogy, and constructive intuition play their part here. But once the correct hypothesis is formulated, the principle of mathematical induction is often sufficient to provide the proof. Inasmuch as such a proof does not give a clue to the act of discovery, it might more fittingly be called a *verification*.
> —Courant and Robbins, *What is Mathematics*, Section I.2.4.

In this section we discuss some ways to guess formulas for sequences. (P.S. Courant and Robbins is an excellent book to add to your personal library.)

5.1 Experimenting With Patterns

Here is a true story of a ninth-grader. He was thinking about the game of Go which is played on a 19×19 grid and was wondering how many intersection points there were on the board. So he wanted to know what 19^2 was, but did not have a piece of paper handy. He wondered if he could figure out how much needed to be subtracted from $20^2 = 400$ to get 19^2. He mentally envisioned the following table:

$$\begin{array}{ccccccccccccccccccc}
0 & & 1 & & 2 & & 3 & & 4 & & 5 & & \cdots & & 19 & & 20 \\
0 & & 1 & & 4 & & 9 & & 16 & & 25 & & \cdots & & ? & & 400 \\
& 1 & & 3 & & 5 & & 7 & & 9 & & \cdots & & & & ? &
\end{array}$$

He saw that the square of $n+1$ was obtained by adding $2n+1$ to the square of n. Thus $19^2 + 39 = 20^2$ so $19^2 = 361$.

Later, when he wrote this down, he recognized that he had simply rediscovered the formula $n^2 + (2n+1) = (n+1)^2$, which he had seen before. But now he began to think of something else: When going from 4^2 to 5^2 the amount added was 9, which just happened to be a perfect square. So $4^2 + 3^2 = 5^2$, which means that $(4, 3, 5)$ is a Pythagorean triple.

Might not other Pythagorean triples be found this way? They can for integer values of n for which $2n + 1$ is a perfect square. Or, working backwards, start with a perfect square m^2, set it equal to $2n + 1$, solve for n, and see if n is an integer. A little reflection convinced him that this works if and only if m is odd. In this case, $n = (m^2 - 1)/2$ and you have the Pythagorean triple $(m, n, n + 1)$.

This gave him a method of generating some Pythagorean triples:

m	n	$n+1$
3	4	5
5	12	13
7	24	25
9	40	41

He noticed that not all Pythagorean triples were generated this way; for example, the triple $(6, 8, 10)$ would be absent. But he realized he could make more triples using similar formulas. For example, he could start with $n^2 + (4n + 4) = (n + 2)^2$. If $4n + 4$ happened to be a perfect square m^2, then he could solve for n, getting $n = (m^2 - 4)/4$ and the triple $(m, n, n + 2)$. He realized that n would be an integer if and only if m were even. So he generated more triples:

4	3	5
6	8	10
8	15	17
10	24	25

Finally, he generalized this procedure by using the formula $n^2 + (2nf + f^2) = (n + f)^2$. If he started with a perfect square m^2, set it equal to $2nf + f^2$, and solved for n, he got $n = (m^2 - f^2)/(2f)$. If n turns out to be an integer, the Pythagorean triple $(m, n, n + f)$ results. By choosing any number m, running through all possibilities of f from 1 to m, he realized that all Pythagorean triples starting with m could be found.

He wrote up this investigation as a science fair project, received the grand prize in his school and an honorable mention in his county.

5.2 Polya's Example

What is the maximum number of regions you can divide space up into using 7 planes? The investigation of this type of problem, especially highlighting techniques of problem solving, is the subject of an old film by George Polya entitled "Let us Teach Guessing," available from the Mathematical Association of America. I saw this film with my class in high school, and it made a profound impression on me.

When thinking about this problem, consider simpler problems (use fewer planes) and analogous problems (dividing a plane up by lines or dividing a line up by points). Try to relate the results.

In my opinion, no teacher should be without Polya's book *How to Solve It*. Another good book of his is *Mathematical Discovery*. Two more advanced books on problem solving by Polya are *Mathematics and Plausible Reasoning, Volume I: Induction and Analogy in Mathematics* and *Mathematics and Plausible Reasoning, Volume II: Patterns of Plausible Inference*.

5.3 Some More Examples

1. What is the next term in the sequence? In each case, think of at least three different "plausible" answers.

   (a) $1, 2, 3, \ldots$.

   (b) $1, 2, 4, \ldots$.

   (c) $1, 2, 5, \ldots$.

2. What is the next term in the sequence? In each case, try to find a formula $f(n)$ such that the sequence is given by $f(0), f(1), f(2), \ldots$.

   (a) $0, 1, 4, 9, 16, 25, \ldots$.

   (b) $0, 0, 6, 24, 60, 120, \ldots$

   (c) $0, 1, 4, 11, 24, 45, \ldots$.

   (d) $2, 7, 30, 125, 508, 2043, \ldots$.

   (e) $1, 1, 3, 5, 11, 21, 43, \ldots$.

3. What is the formula?

   (a) What is the maximum number of pieces into which a pancake can be cut by n straight cuts, each of which crosses each of the others?

   (b) What is the maximum number of pieces that can be produced by n simultaneous straight cuts of a flat figure shaped like a crescent moon?

   (c) How many pieces of cheesecake can be produced by n simultaneous plane cuts of a cylindrical cake?

   (d) Into how many parts can the plane be divided by intersecting circles of the same size? Of different sizes? By intersecting ellipses of different sizes?

(e) Into how many regions can space be divided by intersecting spheres?

(f) With an unlimited supply of toothpicks of n different colors, how many different triangles can be formed on a flat surface, using three toothpicks for the three sides of each triangle? (Reflections are considered different, but not rotations.) How many different squares?

(g) How many different tetrahedra can be produced by coloring each face a solid color and using n different colors? (Two tetrahedra are the same if they can be turned and placed side by side so that corresponding sides match in color.) How many cubes with n colors?

(h) What is the maximum number of pieces that can be produced by n simultaneous plane cuts through a doughnut?

5.4 Finite Differences

Suppose you are asked to find a function $f(x)$ such that

$$\begin{array}{rcr} f(0) & = & -7 \\ f'(0) & = & 5 \\ f''(0) & = & -6 \\ f'''(0) & = & 12 \\ f^{(4)}(0) & = & 0 \\ f^{(5)}(0) & = & 0 \\ f^{(6)}(0) & = & 0 \\ & \vdots & \end{array}$$

We might guess that the function is a polynomial of degree 3. How can we determine the coefficients?

$$\begin{array}{rcl} f(x) & = & c_0 + c_1 x + c_2 x^2 + c_3 x^3 \\ f'(x) & = & c_1 + 2c_2 x + 3c_3 x^2 \\ f''(x) & = & 2c_2 + 6c_3 x \\ f'''(x) & = & 6c_3 \end{array} \qquad \begin{array}{rcl} f(0) & = & c_0 \\ f'(0) & = & c_1 \\ f''(0) & = & 2c_2 \\ f'''(0) & = & 6c_3 \end{array} \qquad \begin{array}{rcl} c_0 & = & f(0) \\ c_1 & = & f'(0) \\ c_2 & = & f''(0)/2 \\ c_3 & = & f'''(0)/6 \end{array}$$

In our example, $c_0 = -7$, $c_1 = 5$, $c_2 = -3$, and $c_4 = 2$ so $f(x) = -7 + 5x - 3x^2 + 2x^3$. In general, if you think $f(x)$ is a polynomial of degree m, $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_m x^m$, then

$$\begin{aligned}
c_0 &= f(0)/0! \\
c_1 &= f'(0)/1! \\
c_2 &= f''(0)/2! \\
&\vdots \\
c_m &= f^{(m)}(0)/m!
\end{aligned}$$

so

$$f(x) = f(0)\frac{x^0}{0!} + f'(0)\frac{x^1}{1!} + f''(0)\frac{x^2}{2!} + \cdots + f^{(m)}(0)\frac{x^m}{m!}.$$

What is the number of different triangles you can form on a flat surface using three toothpicks for the three sides of each triangle, given an unlimited supply of toothpicks of n different colors?

number of colors	number of triangles
0	0
1	1
2	4
3	11
4	24
5	45
$\vdots$	$\vdots$

We want a formula $f(n)$ so that

$$f(0), f(1), f(2), f(3), f(4), f(5), \ldots = 0, 1, 4, 11, 24, 45, \ldots.$$

Look for a pattern by subtracting these numbers from each other, making a difference table:

$$\begin{array}{ccccccccccccc}
0 & & 1 & & 4 & & 11 & & 24 & & 45 & & \cdots \\
& 1 & & 3 & & 7 & & 13 & & 21 & & \cdots & \\
& & 2 & & 4 & & 6 & & 8 & & \cdots & & \\
& & & 2 & & 2 & & 2 & & \cdots & & & \\
& & & & 0 & & 0 & & \cdots & & & &
\end{array}$$

Consider some known formulas:

$$f(n) = n^2 : \quad \begin{array}{ccccccccccccc}
0 & & 1 & & 4 & & 9 & & 16 & & 25 & & \cdots \\
& 1 & & 3 & & 5 & & 7 & & 9 & & \cdots & \\
& & 2 & & 2 & & 2 & & 2 & & \cdots & & \\
& & & 0 & & 0 & & 0 & & \cdots & & &
\end{array}$$

$$
f(n) = n^3 - n: \qquad
\begin{array}{ccccccccccccc}
0 & & 0 & & 6 & & 24 & & 60 & & 120 & & \cdots \\
& 0 & & 6 & & 18 & & 36 & & 60 & & \cdots & \\
& & 6 & & 12 & & 18 & & 24 & & \cdots & & \\
& & & 6 & & 6 & & 6 & & \cdots & & & \\
& & & & 0 & & 0 & & \cdots & & & &
\end{array}
$$

This suggests that for our problem we seek a formula of degree 3.

Let's call the numbers in the first row

$$f(0), f(1), f(2), f(3), f(4), \ldots$$

and the numbers in the second row

$$f'(0), f'(1), f'(2), f'(3), f'(4), \ldots$$

and the numbers in the third row

$$f''(0), f''(1), f''(2), f''(3), f''(4), \ldots$$

etc. These aren't equal to derivatives in the sense of differential calculus, but there seems to be a strong analogy.

$$f'(n) = \frac{f(n+1) - f(n)}{1} \qquad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

In differential calculus, we exploited;

function	derivative
x^0	0
x^1	$1x^0$
x^2	$2x^1$
x^3	$3x^2$
$\vdots$	$\vdots$
x^k	kx^{k-1}

What is the analog for differences? Define

$$[n]^k = \underbrace{n(n-1)(n-2)\cdots(n-k+1)}_{k \text{ terms}}$$

This is sometimes called a falling factorial.

	function	"derivative"	
1	$[n]^0$	0	0
n	$[n]^1$	$1[n]^0$	1
$n(n-1)$	$[n]^2$	$2[n]^1$	$2n$
$n(n-1)(n-2)$	$[n]^3$	$3[n]^2$	$3n(n-1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n(n-1)\cdots(n-k+1)$	$[n]^k$	$k[n]^{k-1}$	$kn(n-1)\cdots(n-k+2)$

Verification:

$$\begin{aligned}
([n]^k)' &= [n+1]^k - [n]^k \\
&= (n+1)(n)(n-1)\cdots(n-k+2) - n(n-1)\cdots(n-k+2)(n-k+1) \\
&= n(n-1)\cdots(n-k+2)\underbrace{((n+1)-(n-k+1))}_{k} \\
&= k[n]^{k-1}.
\end{aligned}$$

Now we can guess formulas:

$$\begin{array}{lcl@{\qquad}lcl@{\qquad}lcl}
f(n) &=& c_0 + c_1[n] + c_2[n]^2 + c_3[n]^3 & f(0) &=& c_0 & c_0 &=& f(0) \\
f'(n) &=& c_1 + 2c_2[n] + 3c_3[n]^2 & f'(0) &=& c_1 & c_1 &=& f'(0) \\
f''(n) &=& 2c_2 + 6c_3[n] & f''(0) &=& 2c_2 & c_2 &=& f''(0)/2 \\
f'''(n) &=& 6c_3 & f'''(0) &=& 6c_3 & c_3 &=& f'''(0)/6
\end{array}$$

In our example,

$$\begin{array}{lcl@{\qquad}lcl}
f(0) &=& 0 & c_0 &=& 0 \\
f'(0) &=& 1 & c_1 &=& 1 \\
f''(0) &=& 2 & c_2 &=& 1 \\
f'''(0) &=& 2 & c_3 &=& 1/3
\end{array}$$

$$\begin{aligned}
f(n) &= 0 + 1[n]^1 + 1[n]^2 + \frac{1}{3}[n]^3 \\
&= n + n(n-1) + \frac{n(n-1)(n-2)}{3} \\
&= \frac{n^3+2n}{3}.
\end{aligned}$$

Now we have a formula that we can try to prove, e.g., by induction, or directly.

In general, fetch $f(0)$, $f'(0)$, $f''(0)$, $\ldots$, $f^{(k)}(0)$ as the first entries of the rows of the difference table (assuming we have reached a row of zeroes). Then

$$\begin{aligned}
c_0 &= f(0)/0! \\
c_1 &= f'(0)/1! \\
c_2 &= f''(0)/2! \\
&\vdots \\
c_k &= f^{(k)}(0)/k!
\end{aligned}$$

$$\begin{aligned}
f(n) &= c_0 + c_1[n] + c_2[n]^2 + \cdots + c_k[n]^k \\
&= f(0)\frac{[n]^0}{0!} + f'(0)\frac{[n]^1}{1!} + f''(0)\frac{[n]^2}{2!} + \cdots + f^{(k)}(0)\frac{[n]^k}{k!}.
\end{aligned}$$

Note that

$$\frac{[n]^j}{j!} = \frac{n(n-1)\cdots(n-j+1)}{j(j-1)\cdots 3\cdot 2\cdot 1} = \binom{n}{j}$$

so

$$f(n) = f(0)\binom{n}{0} + f'(0)\binom{n}{1} + f''(0)\binom{n}{2} + \cdots + f^{(k)}(0)\binom{n}{k},$$

(taking $\binom{n}{k} = 0$ if $n < j$).

In our example,

$$f(n) = 0\binom{n}{0} + 1\binom{n}{1} + 2\binom{n}{2} + 2\binom{n}{3}.$$

Here is another way of doing the same thing—via "antiderivatives."

$$f''''(n) = 0.$$

$$f'''(n) = K.$$

$$f'''(0) = 2 \Longrightarrow 2 = K \Longrightarrow f'''(n) = 2 = 2[n]^0.$$

$$f''(n) = 2[n]^1 + L.$$

$$f''(0) = 2 \Longrightarrow 2 = L \Longrightarrow f''(n) = 2[n]^1 + 2[n]^0.$$

$$f'(n) = [n]^2 + 2[n]^1 + M.$$

$$f'(0) = 1 \Longrightarrow 1 = M \Longrightarrow f'(n) = [n]^2 + 2[n]^1 + [n]^0.$$

$$f(n) = \frac{1}{3}[m]^3 + [n]^2 + [n]^1 + N.$$

$$f(0) = 0 \Longrightarrow 0 = N.$$

$$f(n) = \frac{1}{3}[n]^3 + [n]^2 + [n]^1.$$

and you can verify that this is equal to $2\binom{n}{3} + 2\binom{n}{2} + \binom{n}{1}$, which is the formula we had found before.

5.5 Exponentials

What about the following sequence?

$$\begin{array}{ccccccccccccc}
2 & & 7 & & 30 & & 125 & & 508 & & 2043 & \cdots \\
& 5 & & 23 & & 95 & & 383 & & 1535 & \cdots & \\
& & 18 & & 72 & & 288 & & 1152 & \cdots & &
\end{array}$$

If we take quotients of entries in the last row instead of differences, we see that every quotient is 4. So $f''(n)$ is a geometric, not an arithmetic sequence (as in our earlier examples).

$$f''(n) = 18 \cdot 4^n.$$

What is the "antiderivative" of 4^n? Maybe we can figure this out if we can determined the "derivative" of 4^n.

$$4^{n+1} - 4^n = 4^n(4-1) = 3 \cdot 4^n.$$

So the "antiderivative" of 4^n is $\frac{1}{3}4^n$.

Now we can find a formula for the sequence.

$$f''(n) = 18 \cdot 4^n$$

$$f'(n) = 6 \cdot 4^n + K$$

$$f'(0) = 5 \Longrightarrow 5 = 6 + K \Longrightarrow K = -1 \Longrightarrow f'(n) = 6 \cdot 4^n - [n]^0$$

$$f(n) = 2 \cdot 4^n - [n]^1 + L$$

$$f(0) = 2 \Longrightarrow 2 = 2 + L \Longrightarrow L = 0 \Longrightarrow f(n) = 2 \cdot 4^n - n$$

This method works in general if $f^{(k)}(n)$ is geometric.

5.6 Using Series

Playing around with series can give more techniques. Remember how to figure out the formula for geometric series like

$$h(x) = 1 + 2x + (2x)^2 + (2x)^3 + \cdots$$

$$\begin{aligned} h(x) &= 1 + 2x + (2x)^2 + (2x)^3 + \cdots \\ -2xh(x) &= \quad\; -2x - (2x)^2 - (2x)^3 - \cdots \end{aligned}$$

$$(1 - 2x)h(x) = 1$$

$$h(x) = \frac{1}{1 - 2x} = (1 - 2x)^{-1}$$

We can keep this in mind as we tackle a "Fibonacci"-like sequence:

$$\begin{array}{ccccccccccccccc} 1 & & 1 & & 3 & & 5 & & 11 & & 21 & & 43 & & \cdots \\ & 0 & & 2 & & 2 & & 6 & & 10 & & 22 & & \cdots & \\ & & 2 & & 0 & & 4 & & 4 & & 12 & & \cdots & & \\ & & & -2 & & 4 & & 0 & & 8 & & \cdots & & & \end{array}$$

We never seem to get a row of zeroes. The second row "looks like" twice the first row; i.e., $f(n+1) = f(n) + 2f(n-1)$ for $n \geq 1$. (The ordinary Fibonacci sequence satisfies $f(n+1) = f(n) + f(n-1)$.)

Let define a power series using $f(n)$ as the coefficient of x^n:

$$g(x) = 1 + x + 3x^2 + 5x^3 + 11x^4 + \cdots$$

The relationship $f(n+1) - f(n) - 2f(n-1) = 0$ suggests:

$$\begin{aligned} g(x) &= 1 + x + 3x^2 + 5x^3 + 11x^4 + \cdots \\ -xg(x) &= \quad\;\; -x - x^2 - 3x^3 - 5x^4 - \cdots \\ -2x^2g(x) &= \qquad\quad -2x^2 - 2x^3 - 6x^4 - \cdots \end{aligned}$$

$$(1 - x - 2x^2)g(x) = 1$$

$$g(x) = \frac{1}{-2x^2 - x + 1}$$

Remember, $f(n)$ is the coefficient of x^n. Let's try to find it.

$$g(x) = \frac{1}{-2x^2 - x + 1} = \frac{1}{(1 - 2x)(1 + x)} = \frac{2/3}{1 - 2x} + \frac{1/3}{1 + x}$$

We did the last step using the method of partial fractions. Continuing,

$$= (2/3)(1-2x)^{-1} + (1/3)(1+x)^{-1}$$

$$\begin{array}{l} (2/3)(1+2x+(2x)^2+(2x)^3+\cdots \\ +(1/3)(1-x+x^2-x^3+\cdots \end{array}$$

since we have geometric series.

So the coefficient of x^n is

$$(2/3)2^n + (1/3)(-1)^n$$

and this is our guess for the formula for $f(n)$.

Exercise: Derive a formula for the ordinary Fibonacci sequence this way.

5.7 Using Matrices

Let's look at the previous sequence $1, 1, 3, 5, 11, 21, 43, \ldots$ another way. Make vectors out of pairs of adjacent elements,

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 11 \end{bmatrix} \quad \cdots$$

and find a matrix that transforms each vector into the next, using the "Fibonacci" nature of the sequence.

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

In general,

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} f(n) \\ f(n+1) \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

and calculate A^n by diagonalizing A

$$\begin{array}{cc} \text{eigenvalues} & \text{eigenvectors} \\ 2 & \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ -1 & \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{array}$$

$$A\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{aligned} A &= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}^{-1} \\ &= SDS^{-1} \\ A^n &= (SDS^{-1})(SDS^{-1})\cdots(SDS^{-1}) \\ &= SD^nS^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2^n & 0 \\ 0 & (-1)^n \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2^n & 0 \\ 0 & (-1)^n \end{bmatrix}\begin{bmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix} \end{aligned}$$

So

$$\begin{aligned} A^n\begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2^n & 0 \\ 0 & (-1)^n \end{bmatrix}\begin{bmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} (2/3)2^n + (1/3)(-1)^n \\ (4/3)2^n - (1/3)(-1)^n \end{bmatrix} \\ &= \begin{bmatrix} f(n) \\ f(n+1) \end{bmatrix} \end{aligned}$$

Therefore

$$f(n) = \frac{2}{3}2^n + \frac{1}{3}(-1)^n.$$

Exercise: Try to derive a formula for the ordinary Fibonacci sequence this way.

5.8 Illustrating and Discovering Some Formulas Geometrically

Define

$$S_0(n) = n + 1$$

and

$$S_k(n) = \sum_{i=1}^{n} i^k.$$

We have proved formulas for $S_k(n)$ when k is small, but what about finding formulas for higher values of k? Is there any systematic way to do this?

1. Dissect an $(n+1) \times (n+1)$ square into $(n+1)^2$ smaller squares in the natural way. Now color these smaller squares to justify the following formula geometrically:

   $$(n+1)^2 = 2S_1(n) + S_0(n).$$

   Use this to derive a formula for $S_1(n)$.

2. Dissect an $(n+1) \times (n+1) \times (n+1)$ cube into $(n+1)^3$ small cubes in the natural way. Try to color these smaller cubes to get a formula relating $S_0(n)$, $S_1(n)$, and $S_2(n)$. Use this to derive a formula for $S_2(n)$.

3. Try to guess a generalization of the relationship you discovered from the colorings of the square and the cube.

4. Consider the following list of equations:

   $$\begin{aligned}
   (1+0)^k &= 1 \\
   (1+1)^k &= 1 + \tbinom{k}{1}1^1 + \tbinom{k}{2}1^2 + \cdots + \tbinom{k}{k}1^k \\
   (1+2)^k &= 1 + \tbinom{k}{1}2^1 + \tbinom{k}{2}2^2 + \cdots + \tbinom{k}{k}2^k \\
   (1+3)^k &= 1 + \tbinom{k}{1}3^1 + \tbinom{k}{2}3^2 + \cdots + \tbinom{k}{k}3^k \\
   &\vdots \\
   (1+n)^k &= 1 + \tbinom{k}{1}n^1 + \tbinom{k}{2}n^2 + \cdots + \tbinom{k}{k}n^k
   \end{aligned}$$

   Sum these equations and use this to prove that

   $$(n+1)^k = \sum_{i=0}^{k-1} \binom{k}{i} S_i(n).$$

5. Use the above to find formulas for $S_3(n)$, $S_4(n)$, and $S_5(n)$.

5.9 References and Resources

There are many articles and books on the calculus of finite differences. One nice one is: Martin Gardner, The calculus of finite differences, in *Martin Gardner's New Mathematical Diversions from Scientific American*, Simon and Schuster, New York, 1971, chapter 20.

If you have a sequence and you are trying to figure out what the rule is, or if the sequence is already known elsewhere, the unbeatable place to turn to is the website for the On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.

Many summation formulas can be found with WolframAlpha. For example, type

```
sum i^5, i=1,...,n
```

and the result is

$$\sum_{i=1}^{n} i^5 = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1).$$