Some Problems

Contents

1 Identical

Once there were two girls in my class who looked exactly like each other. I assumed they were identical twins, but when I asked them about this, they said that they weren't. But they claimed to be born on the same day to the same parents. How can this be?

2 Game of 100

1. This is a two player game. Begin with the number zero. Players alternately add a positive integer from 1 to 6, inclusive, to the current running sum. The first player to exactly achieve the number 100 wins. Who can win? Is there a winning strategy?

2. Now analyze the more general situation, when the goal number is the positive number n (instead of 100) and you can add a positive integer from 1 to k, where k is a positive number.

3. Now suppose the goal number is 100 but on your turn you can only add 1, 2, or 5.

3 Rowena's Picture

The fair maiden Rowena wishes to wed. And her father, the evil king, has devised a quiz to drive off her suitors. There are three boxes on a table. One is made of gold, one is made of silver, and the third is made of lead. Inside one of these boxes is a picture of the fair Rowena. It is the job of the knight to figure out which box has her picture. Now, to assist him in this endeavor there are inscriptions on each of the boxes. The gold box says, "Rowena's picture is in this box." The silver box says, "The picture ain't in this box." The lead box says, "The picture ain't in the gold box." If the knight just opens one box and it contains the picture, he gets the girl. The hint is, one of the statements, and only one, is true. The question is: Where's the picture?

4 The Attic Lamp Switch

A downstairs panel contains three on-off switches, one of which controls the lamp in the attic—but which one? Your mission is to do something with the switches, then determine after <u>one</u> trip to the attic which switch is connected to the attic lamp.

5 Penny Piles

This is another two-player game. The initial position is three piles of pennies (or some other object). The first pile has 3 pennies, the second pile has 5 pennies, and the third has 7 pennies. Players alternately take turns, which consist of removing any positive number of pennies from any one pile. The winner is the player who removes the very last penny of them all. What is a good strategy? Who will win?

6 Counterfeit Coins

1. Suppose you have a collection of nine coins, eight of which are of equal weight, and the ninth which is counterfeit and a little bit heavier. How can you use a two-pan balance scale to perform weighings of various piles of the coins to determine which coin is counterfeit? How many weighings are required to guarantee finding the counterfeit coin?

2. Generalize the above problem and its solution to the case when the number of coins is a power of 3.

3. Now suppose you have twelve coins, among which is one counterfeit coin, which may be either a little bit lighter or a little bit heavier. How many weighings will you need to guarantee finding the counterfeit coin?

See the National Library of Virtual Manipulatives, `http://nlvm.usu.edu`, Coin Problem.

7 The Average Traveler

1. There are two towns, conveniently called "A" and "B" joined by a road. A business woman must drive from A to B, conduct some business (what else?) and return to A. She wants her total round trip speed to average 60 miles per hour, but she encountered heavy traffic going to B and averaged only 30 miles per hour on the first half of the trip. What should her average speed be on the return trip to meet her goal?

2. Now answer the question if on the first half of the trip she averaged 50 miles per hour.

3. Now answer the question if on the first half of the trip she averaged 31 miles per hour.

8 The Early Arrival

A commuter is in the habit of arriving at his suburban station each evening exactly at five o'clock. His wife always meets the train and drives him home. One day he takes an earlier train, arriving at the station at four. The weather is pleasant, so instead of telephoning home he starts walking along the route always taken by his wife. They meet somewhere on the way. He gets into the car and they drive home, arriving at their house ten minutes earlier than usual. Assuming that the wife always drives at a constant speed, and that on this occasion she left just in time to meet the five o'clock train, can you determine how long the husband walked before he was picked up?

9 Chickens and Eggs

If a chicken and a half can lay an egg and a half in a day and a half, how long would it take two chickens to lay a dozen eggs?

10 Handshaking

1. At a recent party there were n people, $n \geq 2$. Each person shook hands exactly once with each other person. How many handshakes took place?

2. Now suppose each person shook hands at most once with each other person.

   (a) Explain why there must be two people who shook hands the same number of times.

   (b) Explain why the number of individuals who shook hands an odd number of times must be even.

11 Game of Fifteen

In this two person game a set of nine cards, numbered one to nine, is placed face-up and spread out in the center of a table. Two players alternately select any particular card they wish from the center and place it in front of him/her. As soon as either player has, among all the cards in front of him/her, three cards that sum to 15, that person wins. Who can win this game? Is there an optimal strategy?

12 Double Chess

How can you simultaneously play one game of chess with one chessmaster, and another game of chess with another chessmaster, and be able to lose at most one of the games.

13 Strange Purchase

I went into a hardware store, pointed to an item, and asked, "How much does one of these cost?" "Two dollars" was the answer. Referring to the same item, I asked, "How much for twelve?" The reply was "Four dollars." "OK," I said, "I will buy one hundred twenty three." "That will be six dollars." What was I buying?

14 Animal Colors

Write down a three digit number x in which no digit is zero and the first digit is larger than the last digit. Write this number backwards and subtract the reversal from the original number x. Call the result y. (If y is a two-digit number, put a zero in front of it to regard it as having three digits.) Now reverse y and add the reversal to y. Call the result z. Add up the digits of z. Subtract 14. Use the result to obtain a single letter of the alphabet: 1 is A, 2 is B, etc. Think of a country whose name begins with this letter. Look at the second letter of this country's name and think of an animal that begins with that letter. Think of a color that this animal is likely to have.

15 Ones and Twos

There are five ways to express the number 4 in terms of adding together 1's and 2's: $1+1+1+1$, $1+1+2$, $1+2+1$, $2+1+1$, and $2+2$.

1. In how many ways can you express the number 10?
2. In how many ways can you express the positive integer n?

16 Regions of Space

What is the maximum number of regions that you can create by cutting space with 7 planes?

17 Regions of the Plane

What is the maximum number of regions that you can create by cutting a plane with 7 lines? With n lines?

18 Summing Cubes

From the following pattern, state and prove a formula:

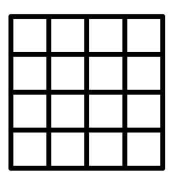
$$\begin{aligned} 1^3 &= 1^2 \\ 1^3 + 2^3 &= (1+2)^2 \\ 1^3 + 2^3 + 3^3 &= (1+2+3)^2 \end{aligned}$$

19 Two-Cube Calendar

I once saw in a store an unusual desk calendar consisting of two cubes that could be arranged side-by-side in a frame. The date was indicated simply by arranging the two cubes so that their front faces gave the date. The face of each cube bore a single digit, 0 through 9, and one could arrange the cubes so their front faces indicated any date from 01, 02, 03, . . . , to 31. How should the faces of the two cubes be labeled so that all the dates can be displayed? (I confess there is a little "twist" to the solution.)

20 Squares in a Grid

Draw an $n \times n$ grid of 1×1 squares. How many total squares of all sizes (1×1, 2×2, etc.) are in this grid?



21 Summing Odds

From the following pattern, state and prove a formula:

$$\begin{aligned} 1 &= 1^2 \\ 1+3 &= 2^2 \\ 1+3+5 &= 3^2 \\ 1+3+5+7 &= 4^2 \end{aligned}$$

22 Summing Odd Squares

Find and justify a closed formula for the sums of the squares of the first n positive odd integers.

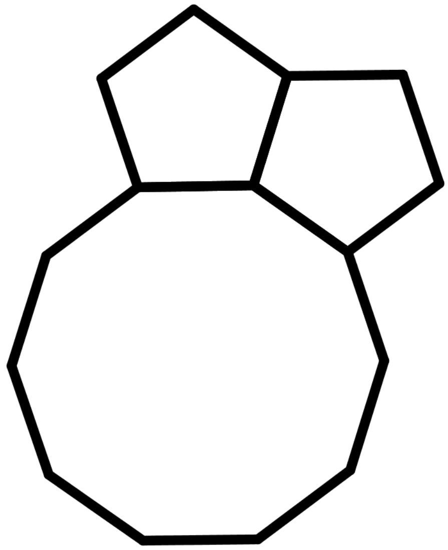
23 Fibonacci Formula

The Fibonacci sequence is defined by $F_1 = 1$, $F_2 = 1$, and $F_k = F_{k-2} + F_{k-1}$ for all integer $k \geq 3$. Prove that

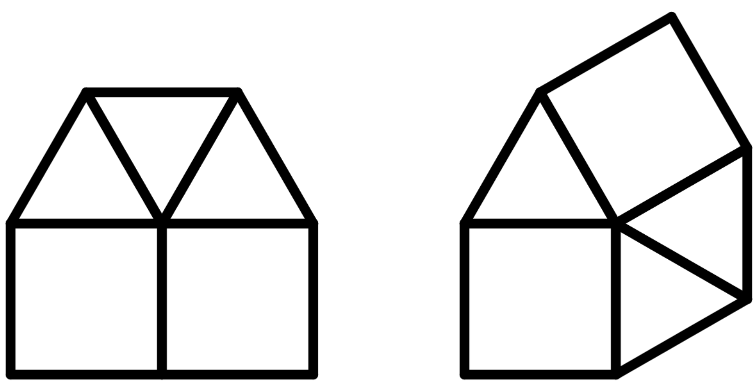
$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

24 Planar Clusters

Consider the problem of planar clusters of regular polygons fitting perfectly together around a single vertex. One regular decagon and two regular pentagons can fit together perfectly in the plane surrounding a common vertex (since the interior angle of a regular decagon measures 144 degrees, and the interior angle of a regular pentagon measures 108 degrees). Let's call this a (10,5,5) cluster (or a (5,10,5) cluster, or a (5,5,10) cluster).



Similarly, two squares and three equilateral triangles can fit together perfectly surrounding a common vertex. There are essentially two different ways to do this: (3,3,3,4,4) (where the squares are adjacent) and (3,3,4,3,4) (where the squares are not adjacent), and we will regard these as two <u>different</u> clusters.



Note that we could have called this last cluster (4,3,3,4,3) as well—it still refers to the same cluster. However, (3,3,3,4,4) and (3,3,4,3,4) are <u>not</u> the same.

1. What is the minimum number of regular polygons that can form a planar cluster?

2. What is the maximum number of regular polygons that can form a planar cluster?

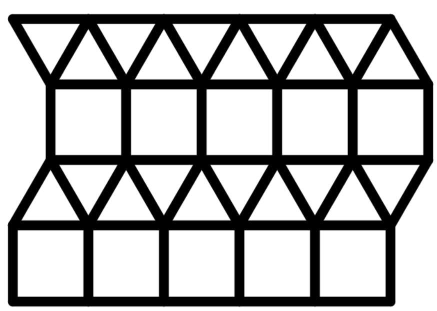
3. You have now seen at least three planar clusters. Determine all possible planar clusters that can be formed by fitting together a combination of three regular polygons in the plane surrounding a common vertex. Be systematic in some fashion, so that you can be certain you have found all of them, and explain clearly how you know this.

4. Explain why the previous problem is equivalent to finding combinations of positive integers $a, b, c \geq 3$ such that
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}.$$

5. Determine all possible planar clusters that can be formed by fitting together a combination of more than three regular polygons in the plane surrounding a common vertex.

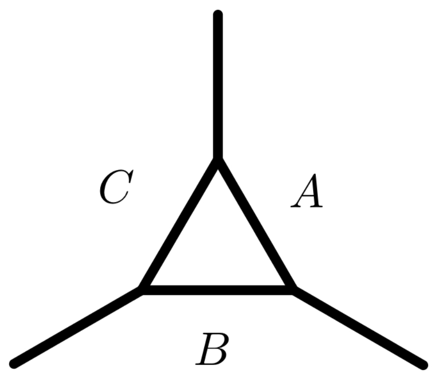
25 Regular and Semiregular Tilings

Some of the clusters in the previous problem can be extended to tile the plane so that at every vertex exactly the same cluster appears—the same sequence of polygons, in either clockwise or counterclockwise order. For example, if you extend the (4,4,4,4) cluster, you get the familiar tiling of the plane with squares, with four squares meeting at each vertex, and if you extend the (3,3,3,4,4) cluster, you get the following tiling of the plane:



1. Of the clusters you have found, determine which ones can be extended to tilings of the plane. Make a precise drawing of each of the tilings that you find.

2. Prove that the (3,10,15) cluster cannot be extended to tile the plane. You may wish to consider the following diagram. If polygon A is a 15-gon, what must polygon B be? How about polygon C?



3. Choose another planar cluster with three polygons meeting at the common vertex that cannot be extended to a tiling, and develop a proof that the extension is not possible.

4. Choose a planar cluster with four polygons meeting at the common vertex that cannot be extended to a tiling, and develop a proof that the extension is not possible.

26 Number Guessing Cards

Here are two versions of the Number Guessing Cards. One person chooses a number that appears on at least one card, then indicates on which cards the chosen number appears. The other person then must as quickly as possible determine which number was chosen.

Small Set

CARD 1	**CARD 2**	**CARD 3**
1	2	4
3	3	5
5	6	6
7	7	7

Large Set

CARD 1	CARD 2	CARD 3	CARD 4	CARD 5
1	2	4	8	16
3	3	5	9	17
5	6	6	10	18
7	7	7	11	19
9	10	12	12	20
11	11	13	13	21
13	14	14	14	22
15	15	15	15	23
17	18	20	24	24
19	19	21	25	25
21	22	22	26	26
23	23	23	27	27
25	26	28	28	28
27	27	29	29	29
29	30	30	30	30
31	31	31	31	31

27 How Many Questions?

You are going to play a game with a friend. Your friend is going to think of a whole number in the range 0 to 127, inclusive. You are going to ask "yes" or "no" questions to determine the number. What is the fewest number of questions that you can ask to eventually determine the number?

28 Prime Numbers

Find all the prime numbers between 1 and 100.

29 Weird Multiplication I

Why does the following method work?

Suppose you want to calculate 18×14. Write these two numbers at the top of two columns. In the first column, start with 18, divide it by two, and repeat, ignoring any remainders, until you reach the number 1. In the second column, start with 14, double it, and repeat, until you have a column of equal length. Draw a line through any number in the first column that is even, and extend the line to cross out the neighboring number in the second column. Add up the remaining numbers in the second column. This will be the product, and this procedure works for any pair of numbers. Two examples:

$$\begin{array}{rr} \cancel{18} & \cancel{14} \\ 9 & 28 \\ \cancel{4} & \cancel{56} \\ \cancel{2} & \cancel{112} \\ 1 & 224 \\ \hline & 252 \end{array} \qquad \begin{array}{rr} 13 & 25 \\ \cancel{6} & \cancel{50} \\ 3 & 100 \\ 1 & 200 \\ \hline & 325 \end{array}$$

30 Weird Multiplication II

Why does the following method work?

Number the fingers on each hand 5, 6, 7, 8, and 9, starting with your pinkies and ending with your thumbs. Hold your hands out in front of you with the palms facing you, thumbs up, and pinkies down.

To multiply two numbers on the above list together, touch the corresponding fingers together. For example, to calculate 8×7, touch the index finger of your left hand to the middle finger of your right hand.

At and above the index finger on your left hand there are 2 fingers. At and above the middle finger on your right hand there are 3 fingers. Multiply these two numbers together: $2 \times 3 = 6$. The remaining fingers on both hands combined each count 10, and there are 5 of them, for a product of 50. Add 6 and 50 to get the answer: $8 \times 7 = 56$.

Confused? Let's try another one: 6×8. Touch the ring finger of your left hand to the index finger of your right hand. Multiplying the number of fingers at and above the touching fingers: $4 \times 2 = 8$. Counting fingers below the touching fingers as tens: 40. Add 8 and 40 to get the answer: 48.

31 Binary Numbers

When we express numbers in the base 10 system, we use the symbols 0 through 9 as digits. The places of the digits are associated with powers of 10. The rightmost digit is in the 1's place, and moving left we then have the 10's place, the 100's place, the 1000's place, and so on. For example, the numerals 4253 represent the number 4 thousands, 2 hundreds, 0 tens, and 3 ones.

When we express numbers in the base 2 (or binary) system, we use the symbols 0 and 1 as digits. The places of the digits are associated with powers of 2. The rightmost digit is in the 1's place, and moving left we then have the 2's place, the 4's place, the 8's place, and so on. For example, the numerals 1101 represent the number 1 eight, 1 four, 0 twos, and 1 one (which is the number 13 when written in base ten).

1. Make a table expressing each of the whole numbers from 0 to 31 in base 2.

n	16	8	4	2	1
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	1	0
3	0	0	0	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

2. Study this table and then explain how the Number Guessing Cards work.

3. Prove that no nonnegative integer can have two different binary representations.

4. What does the following number in binary notation equal: $0.\overline{1}$?

5. Prove that you can get the binary representation of a positive integer n in the following way

   (a) Set $i = 0$ and $n_0 = n$.

   (b) Divide n_i by 2, with quotient n_{i+1} and remainder r_i.

   (c) If $n_{i+1} > 0$, increase i by 1 and return to the previous step. Otherwise, go to the next step.

   (d) The decimal representation of n is $r_i r_{i-1} \cdots r_0$.

32 27 Card Trick

This trick is described by Eric Shrader on Cut-the-Knot,
www.cut-the-knot.org/arithmetic/rapid/CardTrick.shtml.

"My grandfather taught me this simple trick when I was young:

"Deal out 27 cards face up into a grid of 9 rows and 3 columns. Do this by dealing out 3 cards horizontally in a row, then 3 more cards just below the first 3, then 3 more, etc., until you have 9 rows. (It's best to overlap cards in a column so that the columns aren't so long; just make sure that the values of all cards are visible.) Discard the remaining cards. Only these 27 will be used to play.

"Ask a spectator to mentally pick a card and remember it. Ask him to tell you only which of the 3 columns it is in.

"Collect the 27 cards into a deck. Gather them vertically such that the column containing the spectator's card is second. Pick up cards from the top of the column to the bottom, keeping them in the same order. For example, if the spectator tells you column 1, then first gather column 2 or 3, then column 1, and then the remaining column. When you're done, the top card of the deck should now be the top card of the column you collected first, followed by the rest of that column in order. Then the 10th card of the deck will be the card at the top of the column containing the spectator's card, etc. If you overlapped the cards as suggested, gathering and keeping them in the correct order is easy.

"Now deal the cards again. Deal exactly as described above—horizontally across the rows first.

"Again, ask him to tell you only which column contains the card.

"Pick up the columns vertically—exactly as above—making sure that the column containing the card is picked up second.

"Finally deal them out a third time in exactly the same way, ask which column contains the card, and gather them up in the same manner.

"The spectator's card will now be the 14th one in the deck. To add drama, I usually deal out the cards one at a time face down. I don't make it obvious that I'm counting and I don't look at any of the cards. Instead, I'll hesitate over certain cards, pretending to get a "vibe" from them, and then I'll finally settle on the right one.

"This seemed like pure magic when I first learned it. Only when I got older did I realize that it's actually simple math. Do you see how it works?"

33 8 Card Trick

This is a variant of the previous trick. This time use 8 cards (though it can be modified for more cards) and two columns. The magician asks one spectator to name (aloud) a whole number N from 0 to 7, and a second spectator to select a card. Cards are dealt and collected three times as before, but now, at the very end after the final gathering, the magician is able to count off precisely N cards, revealing the chosen card as the next one. By what method does the magician stack the columns to force the chosen card to the desired position?

34 South from Key West

If you fly due south out of Key West, Florida, which South American country will you hit first?

35 East of Reno

What's the biggest city in the US east of Reno, Nevada and west of Denver, Colorado?

36 Hiker

A hiker left her camp, hiked 10 miles south, then 10 miles east, ate lunch, then hiked 10 miles north and was back at her camp. Where was her camp?

37 Venn Diagrams

What does a Venn diagram with three circles have to do with writing the numbers from 0 to 7 in base two?

38 Watching TV

In a suburban home live Abner, his wife Beryl and their three children, Cleo, Dale and Ellsworth. The time is 8 p.m. on a winter evening.

1. If Abner is watching television, so is his wife.
2. Either Dale or Ellsworth, or both of them, are watching television.
3. Either Beryl or Cleo, but not both, is watching television.
4. Dale and Cleo are either both watching or both not watching television.
5. If Ellsworth is watching television, then Abner and Dale are also watching.

Who is watching television and who is not?

39 Logical Implications in Algebraic Reasoning

1. Solve $x^2 = 25$.
2. Solve $x^2 < 4$.
3. Solve $x^2 > 5$.
4. Solve $|x-3| > 5$.
5. Solve $|x+1| \leq 6$.
6. Solve $|x+1| + |x-1| = 2$.
7. Solve $\frac{1}{x^2-1} = \frac{1}{3x+3}$.
8. Solve $\frac{x^2}{x-1} = \frac{2-x}{x-1}$.
9. Solve $\frac{1}{\sqrt{x^2-1}} \geq \frac{1}{\sqrt{3x+3}}$.
10. Solve $x(2x+3) = x(x-5)$.
11. Solve $\frac{1}{\frac{1}{x}} = x$.
12. Solve $\sqrt{x^2-5x+5} = \sqrt{x-3}$.

40 Outdoor Barbecue

Tom, John, Fred, and Bill are friends whose occupations are (in no particular order) nurse, secretary, teacher, and pilot. They attended a picnic recently, and each one brought his favorite meat (hamburger, chicken, steak, and hot dogs) to barbecue. From the clues below, determine each man's occupation and favorite meat.

1. Tom is neither the nurse nor the teacher.
2. Fred and the pilot play in a jazz band together.
3. The burger lover and the teacher are not musically inclined.
4. Tom brought hot dogs.
5. Bill sat next to the burger fan and across from the steak lover.
6. The secretary does not play an instrument or sing.

41 Smith, Jones, and Robinson

The following seven facts are given:

1. Smith, Jones and Robinson are the engineer, brakeman and fireman on a train, but not necessarily in that order. Riding the train are three passengers with the same three surnames, to be identified in the following premises by a "Mr." before their names.

2. Mr. Robinson lives in Los Angeles.

3. The brakeman lives in Omaha.

4. Mr. Jones long ago forgot all the algebra he learned in high school.

5. The passenger whose name is the same as the brakeman's lives in Chicago.

6. The brakeman and one of the passengers, a distinguished mathematical physicist, attend the same church.

7. Smith beat the fireman at billiards.

Who is the engineer?

.

42 Five Houses

There are five houses, each a different color and inhabited by men of different nationalities with different pets, drinks, and favorite foods.

1. The Englishman lives in the red house.

2. The Spaniard owns the dog.

3. Coffee is drunk in the green house.

4. The Ukrainian drinks tea.

5. The green house is immediately to your right of the white house.

6. The pizza eater owns snails.

7. Sushi is eaten in the yellow house.

8. Milk is drunk in the middle house.
9. The Norwegian lives in the first house on the left.
10. The man who eats burritos lives next to the man with the fox.
11. Sushi is eaten next to the house with a horse.
12. The steak eater drinks orange juice.
13. The Japanese eats jambalaya.
14. The Norwegian lives next to the blue house.

Who drinks water and who owns the zebra?

43 Logic Problems

For each of the following statements, determine which combinations of T (true) and F (false) are consistent with the statement.

1. A or B

A	B
F	F
F	T
T	F
T	T

2. A and B

A	B
F	F
F	T
T	F
T	T

3. If A then B

A	B
F	F
F	T
T	F
T	T

4. A if and only if B

A	B
F	F
F	T
T	F
T	T

5. Not A

A	B
F	F
F	T
T	F
T	T

6. Not (A or B)

A	B
F	F
F	T
T	F
T	T

7. Not (A and B)

A	B
F	F
F	T
T	F
T	T

8. Not (if A then B)

A	B
F	F
F	T
T	F
T	T

9. Not (A if and only if B)

A	B
F	F
F	T
T	F
T	T

10. (A or B) or C

A	B	C
F	F	F
F	F	T
F	T	F
F	T	T
T	F	F
T	F	T
T	T	F
T	T	T

11. (A or B) and C

A	B	C
F	F	F
F	F	T
F	T	F
F	T	T
T	F	F
T	F	T
T	T	F
T	T	T

12. If (A or B) then C

A	B	C
F	F	F
F	F	T
F	T	F
F	T	T
T	F	F
T	F	T
T	T	F
T	T	T

13. If C then (A or B)

A	B	C
F	F	F
F	F	T
F	T	F
F	T	T
T	F	F
T	F	T
T	T	F
T	T	T

14. (A or B) if and only if C

A	B	C
F	F	F
F	F	T
F	T	F
F	T	T
T	F	F
T	F	T
T	T	F
T	T	T

15. (A and B) or C

A	B	C
F	F	F
F	F	T
F	T	F
F	T	T
T	F	F
T	F	T
T	T	F
T	T	T

16. (A and B) and C

A	B	C
F	F	F
F	F	T
F	T	F
F	T	T
T	F	F
T	F	T
T	T	F
T	T	T

17. If (A and B) then C

A	B	C
F	F	F
F	F	T
F	T	F
F	T	T
T	F	F
T	F	T
T	T	F
T	T	T

18. If C then (A and B)

A	B	C
F	F	F
F	F	T
F	T	F
F	T	T
T	F	F
T	F	T
T	T	F
T	T	T

19. (A and B) if and only if C

A	B	C
F	F	F
F	F	T
F	T	F
F	T	T
T	F	F
T	F	T
T	T	F
T	T	T

44 Red and Green Hats

Three students — Alfred, Beth, and Carla — are blindfolded and told that either a red or a green hat will be placed on each of them. After this is done, the blindfolds are removed; the students are asked to raise a hand if they see a red hat, and to leave the room as soon as they are sure of the color of their own hat. All three hats happen to be red, so all three students raise a hand. Several minutes go by until Carla, who is more astute than the others, leaves the room. How did she deduce the color of her hat?

45 Consecutive Numbers

Two students, A and B, are chosen from a math class of highly logical individuals. They are each given one positive integer. Each knows his/her own number, and is trying to determine the other's number. They are informed that their numbers are consecutive. In each of the following scenarios, what can you deduce about the two numbers?

1. First Scenario

   A: I know your number.

   B: I know your number.

2. Second Scenario

   A: I don't know your number.

   B: I know your number.

   A: I know your number.

3. Third Scenario

   A: I don't know your number.

   B: I don't know your number.

   A: I know your number.

   B: I know your number.

4. Fourth Scenario

   A: I don't know your number.

   B: I don't know your number.

   A: I don't know your number.

   B: I know your number.

   A: I know your number.

5. Fifth Scenario

   A: I don't know your number.

   B: I don't know your number.

   A: I don't know your number.

   B: I don't know your number.

   A: I know your number.

   B: I know your number.

46 How Many Children?

"I hear some youngsters playing in the back yard," said Jones, a graduate student in mathematics. "Are they all yours?"

"Heavens, no," exclaimed Professor Smith, the eminent number theorist. "My children are playing with friends from three other families in the neighborhood, although our family happens to be largest. The Browns have a smaller number of children, the Greens have a still smaller number, and the Blacks the smallest of all."

"How many children are there altogether?" asked Jones.

"Let me put it this way," said Smith. "There are fewer than 18 children, and the product of the numbers in the four families happens to be my house number which you saw when you arrived."

Jones took a notebook and pencil from his pocket and started scribbling. A moment later he looked up and said, "I need more information. Is there more than one child in the Black family?"

As soon as Smith replied, Jones smiled and correctly stated the number of children in each family.

Knowing the house number and whether the Blacks had more than one child, Jones found the problem trivial. it is a remarkable fact, however, that the number of children in each family can be determined solely on the basis of the information given above!

47 Piano Tuning

A geometric sequence is a sequence of the form $a, ar, ar^2, ar^3, \ldots$. That is to say, the first term is some number a, and thereafter each term is obtained from the previous one by multiplying by a specific number r. For example, $15, 45, 135, 405, \ldots$ is a geometric sequence.

The 88 keys of a piano are tuned according to a geometric sequence. The note A (designated A4) above middle C (designated C4) has frequency 440 Hz. The C note (C5) above middle C has twice the frequency of middle C. There are 13 notes between C4 and C5, inclusive. Determine the frequencies of these 13 notes, explaining your reasoning. Round your answers to two decimal places.

1. C (C4)
2. C♯
3. D
4. D♯
5. E
6. F
7. F♯
8. G
9. G♯

10. A 440 Hz

11. A$\sharp$

12. B

13. C (C5)

To the human ear, two notes played together tend to sound more harmonious if the ratios of their frequencies is a ratio of two small integers. For each of the fractions below, find a note above C4 so that the ratio of the frequency of this note to the frequency of C4 is approximately equal to that fraction:

1. 2/1
2. 3/2
3. 4/3
4. 5/4
5. 5/3
6. 6/5

Here is a website with a virtual keyboard to test notes and chords: `www.bgfl.org/bgfl/custom/resources_ftp/client_ftp/ks2/music/piano/index.htm`.

48 Guitar Frets

1. Mathematically determine the locations of the first 12 frets on a guitar.
   - Measure the length of a string, say, the low E.
   - The next twelve notes will be F, F#, G, G#, A, A#, B, C, C#, D, D#, and E.
   - The frequency of the upper E is twice the frequency of the lower E.
   - The frequency of each successive note is a certain constant, a, times the frequency of the preceding note.
   - The frequency of a note is inversely proportional to the length of the part of the string that is vibrating.

2. One of your classmates observed that the spacing between two successive pairs of frets decreases by a constant ratio. Is this true? What is this ratio?

3. For each of the following fractions, find a note such that the ratio of the frequency of this note to the frequency of the low E is approximately equal to this fraction.

   (a) 2/1
   (b) 3/2
   (c) 4/3
   (d) 5/4
   (e) 6/5

49 Sum of Two Primes

Find all the ways to write the number 543 as the sum of two primes.

50 Model T

In 1913 the Model T automobile cost $550. Use the Consumer Price Index to express this amount in 2013 dollars. Was this car expensive or not? What is an estimate of the annual inflation rate from 1913 to 2013? See `http://www.bls.gov/data/inflation_calculator.htm`.

51 Interesting Problems

1. A dealer offers to sell you a car for $25,432. The claim is that this is a 10% reduction off the original price. What was the original price?

2. Sales are booming in the toy department, and an especially popular toy's price was increased by 7% to the new price of $31.59. What was the original price?

52 Savings

A newborn's parents set up a college fund. They plan to invest $100 on the first day of each month. If the fund pays 6% annual interest, compounded monthly on the last day of each month, what is the future value of the fund in 18 years; i.e., at the end of month 216?

53 Loans

You receive a loan of \$25,000, at an annual interest rate of 6% compounded monthly. You will repay the loan by making equal monthly payments at the end of each month over 36 months. How much will each payment be?

54 Credit Cards

Look at the terms of a particular credit card here: `http://www.ms.uky.edu/~lee/ma310sp13/creditcard.pdf`. Suppose you buy a TV set and charge \$1000 to this card. Then you faithfully make the minimum required payment each month. How many months will it take you to pay this off? What is the total amount you will end up paying? What happens if you miss a payment? For the purposes of this problem, since we are not making more purchases throughout the period, let's assume that interest is compounded monthly instead of daily. That is to say, at the end of each month, interest on the balance is calculated using the (decimal) rate of 0.1999/12.

55 Weighty Questions

1. Which weighs more, a pound of lead or a pound of feathers?
2. Which weighs more, a pound of gold or a pound of feathers?

56 Partial Sums of Geometric Series

1. Find and justify a formula for $a + ar + ar^2 + \cdots + ar^n$.
2. Convert $0.\overline{123}$ to a rational expression.
3. How are these two problems related?

57 General Periodic Savings Formula

Suppose you deposit an amount A at the beginning of each of m time periods, and interest at the time period rate of r (expressed as a decimal) is added to the current balance at the end of each time period. (So, for example, if the annual rate is 5% and the time period is one month, then $r = .05/12$.) Let B be the amount that you will have at the end of m time periods. Prove that

$$B = \frac{A(1+r)[(1+r)^m - 1]}{r}.$$

58 General Loan Formula

Suppose you borrow an amount B and repay it by paying an amount A at the end of each of m time periods. Assume that interest is charged at the time period rate of r (expressed as a decimal). (So, for example, if the annual rate is 5% and the time period is one month, then $r = .05/12$.) Prove that

$$B = \frac{A[(1+r)^m - 1]}{r(1+r)^m}.$$

59 Finite Figure Symmetries

A finite figure is a bounded subset of the plane. A symmetry of a finite figure is a rotation (by a nontrivial angle) or reflection that maps the figure bijectively onto itself. A finite figure has symmetry type Z_k, $k \geq 1$, if there is no reflectional symmetry, there is a rotational symmetry by an angle $360/k$, and there is no smaller angle of rotational symmetry. A finite figure has symmetry type D_k, $k \geq 1$, if there are exactly k lines of reflectional symmetry. Practice classifying the symmetry types of finite figures, including those found around you.

60 Border Pattern Symmetries

A subset of the plane has translational symmetry if there is a translation (by a nontrivial amount) that maps the figure bijectively onto itself. A border pattern is a subset of the plane for which there is a translation T such that all translational symmetries of the pattern are of the form T^j, $j \in \mathbf{Z}$. That is to say, there is a smallest translational symmetry that generates all translational symmetries.

1. Border patterns may also possess other symmetries: 180 degree rotational, vertical and/or horizontal reflectional, and glide reflectional. Prove that there are only seven possible types of border patterns.

2. Practice classifying the symmetry types of border patterns, including those found around you.

61 Wallpaper Pattern Symmetries

A wallpaper pattern is a subset of the plane for which there are two translations T_1 and T_2 in nonparallel directions such that all translational symmetries of the pattern are of the form $T_1^j T_2^k$, $j, k \in \mathbf{Z}$. That is to say, there are two translational symmetries in two nonparallel directions that together generate all translational symmetries. Wallpaper patterns may also possess other symmetries, and there are 17 different types. Practice classifying the symmetry types of border patterns, including those found around you.

62 Symmetries

1. What is the net effect of reflecting across line ℓ_1 and then reflecting across a perpendicular line ℓ_2?
2. What is the net effect of reflecting across line ℓ_1 and then glide reflecting across a perpendicular line ℓ_2?
3. What is the net effect of reflecting across line ℓ_1 and then reflecting across an intersecting line ℓ_2?
4. What is the net effect of reflecting across line ℓ_1 and then reflecting across a parallel line ℓ_2?
5. What is the net effect of rotating 180 degrees about a point, and then rotating 180 degrees about a different point?
6. What is the net effect of rotating α degrees about a point, and then rotating $-\alpha$ degrees about a different point?
7. What is the net effect of rotating α degrees about a point, and then rotating β degrees about a different point?
8. What is the net effect of reflecting across a line, and then rotating 180 degrees about a point on that line?
9. What is the net effect of reflecting across a line, and then rotating 180 degrees about a point not on that line?
10. What is the net effect of glide reflecting across a line, and then rotating 180 degrees about a point on that line?

63 Complex Numbers and Transformations

In the following we are considering functions of complex numbers.

1. What geometric transformation of the plane is described by $f(z) = -z$?
2. What geometric transformation of the plane is described by $f(z) = iz$?
3. What geometric transformation of the plane is described by $f(z) = \overline{z}$ (the complex conjugate of z)?
4. Let w be a fixed complex number. What geometric transformation of the plane is described by $f(z) = z + w$?
5. Design a complex function that performs a glide reflection across the x-axis with translation by the vector $(2, 0)$.
6. Design a complex function that performs a glide reflection across the x-axis with translation by the vector $(a, 0)$, where a is a real number.
7. Design a complex function that reflects the plane across the y-axis.
8. Design a complex function that reflects the plane across the line $y = 2$.
9. Design a complex function that reflects the plane across the line $y = a$, where a is a real number.
10. Design a complex function that reflects the plane across the line $x = 2$.
11. Design a complex function that reflects the plane across the line $x = a$, where a is a real number.
12. Design a complex function that rotates the plane 180 degrees about the point $(2, 0)$.
13. Design a complex function that rotates the plane 180 degrees about the point $(a, 0)$, where a is a real number.
14. Let w be the fixed complex number $\cos\alpha + i\sin\alpha$. Prove that the complex function $f(z) = wz$ is a counterclockwise rotation of the plane about the origin by the angle α. Suggestion: Write the complex number w in the form $r(\cos\beta + i\sin\beta)$.
15. Design a complex function that rotates the plane by the angle α about the point (a, b).

16. Design a complex function that reflects the plane about the line $ax + by = 0$.

17. Design a complex function that reflects the plane about the line $ax + by = c$.

64 Composing Transformations

Now use compositions of complex functions to answer the following questions (most of which are relevant to thinking about the symmetries of border patterns).

1. What is the net effect of reflecting across the line $x = a$ and then reflecting across the perpendicular line $y = 0$?

2. What is the net effect of reflecting across line $x = a$ and then glide reflecting across the perpendicular line $y = 0$ with translation vector $(b, 0)$?

3. What is the net effect of reflecting across line $x = a$ and then reflecting across a parallel line $x = b$?

4. What is the net effect of rotating 180 degrees about the point $(a, 0)$, and then rotating 180 degrees about the point $(b, 0)$?

5. What is the net effect of rotating α degrees about a point, and then rotating $-\alpha$ degrees about a different point?

6. What is the net effect of rotating α degrees about a point, and then rotating β degrees about a different point?

7. What is the net effect of reflecting across the line $x = a$ and then rotating 180 degrees about the point $(a, 0)$?

8. What is the net effect of reflecting across the line $x = a$ and then rotating 180 degrees about the point $(b, 0)$?

9. What is the net effect of reflecting across the line $y = 0$ and then rotating 180 degrees about the point $(a, 0)$?

10. What is the net effect of glide reflecting across the line $y = 0$ with translation vector $(a, 0)$, and then rotating 180 degrees about the point $(b, 0)$?

65 Position and Velocity

1. In each of the following cases, make a graph of velocity as a function of time, describe what kind of motion is taking place, and what the net change in position is. Assume that position is measured in feet and time is measured in seconds. Recall also the differences between distance traveled and change in position, and speed and velocity.

   (a) $v(t) = 0$, $0 \le t \le 5$.

   (b) $v(t) = 10$, $0 \le t \le 5$.

   (c) $v(t) = -10$, $0 \le t \le 5$.

   (d) $v(t) = 10$, $0 \le t < 5$, and $v(t) = -10$, $5 \le t \le 10$.

   (e) $v(t) = c$, $a \le t \le b$.

   (f) In each of the above cases, relate the net change in position to an area.

2. Use the model of a person walking on a train to model the motion given by the sum of two velocity functions, $v_1(t)+v_2(t)$. How can you determine the net change in position of the person on the train with respect to the ground from the net change of position of the person on the ground with velocity v_1 and the net change of position of the train with velocity v_2?

3. Use the above model with $v_1(t) = 2t$ and $v_2(t) = 10 - 2t$, $0 \le t \le 5$, to explain why if a person is traveling on the ground with velocity $v_1(t)$, the net change in position is given by the area of the triangle under the graph of $v_1(t)$.

4. Use the above model to make sense of the net change in position of a person moving with velocity function $v(t) = kt + c$, $a \le t \le b$.

5. Now consider various graphs of position as a function of time, describe what kind of motion is taking place, and what the velocity function is. Again, assume that position is measured in feet and time is measured in seconds.

   (a) $s(t) = 0$, $a \le t \le b$.

   (b) $s(t) = k$, $a \le t \le b$.

   (c) $s(t) = kt$, $a \le t \le b$.

   (d) $s(t) = kt + c$, $a \le t \le b$.

   (e) In each of the above cases, relate the net change in position to a slope.

66 Rates of Change

1. Explain why the following definition makes sense geometrically:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

2. Argue informally why the following statement makes sense geometrically:

   Let f be a continuous function defined on a closed interval $[a, b]$. Let F be the function defined by

$$F(x) = \int_a^x f(t)dt$$

   for $x \in [a, b]$. Then $F'(x) = f(x)$ for all $x \in (a, b)$.

67 Good Questions

Here are some questions from Cornell's Good Questions Project, `http://www.math.cornell.edu/~GoodQuestions/GoodQuestionSlides.pdf`.