Some Problems

Contents

1 Identical

Once there were two girls in my class who looked exactly like each other. I assumed they were identical twins, but when I asked them about this, they said that they weren't. But they claimed to be born on the same day to the same parents. How can this be?

2 Game of Fifteen

In this two person game a deck of nine cards, numbered one to nine, is placed face-up and spread out in the center of a table. Two players alternately select any particular card they wish from the center and place it face-up in front of him/her. A *set* is a collection of exactly three cards that sum to 15. As soon as either player has a set among all the cards in front of him/her, that person wins. Who can win this game? Is there an optimal strategy?

(From Martin Gardner)

3 Game of 100

This is a two player game. Begin with the number zero. Players alternately add a positive integer from 1 to 6, inclusive, to the current running sum. The first player to exactly achieve the number 100 wins. Who can win? Is there a winning strategy?

4 Variations on Game of 100

Consider these variations on the previous game.

1. Analyze the more general situation, when the goal number is the positive integer n (instead of 100) and you can add a positive integer from 1 to k, where k is a positive number.

2. Suppose the goal number is 100 but on your turn you can only add 1, 2, or 6.

3. Suppose the goal number is the positive integer n and you can add either one of the distinct given positive integers 1 or a.

4. Suppose the goal number is the positive integer n and you can add either one of the distinct given positive integers 1, a, or b.

5 Strange Purchase

I went into a hardware store, pointed to an item, and asked, “How much does one of these cost?” “Two dollars” was the answer. Referring to the same item, I asked, “How much for twelve?” The reply was “Four dollars.” “OK,” I said, “I will buy one hundred twenty three.” “That will be six dollars.” What was I buying?

6 Penny Piles

This is another two-player game. The initial position is three piles of pennies (or some other object). The first pile has 3 pennies, the second pile has 5 pennies, and the third has 7 pennies. Players alternately take turns, which consist of removing any positive number of pennies from any one pile. The winner is the player who removes the very last penny of them all. What is a good strategy? Who will win?

7 How Many Children?

“I hear some youngsters playing in the back yard,” said Jones, a graduate student in mathematics. “Are they all yours?”

“Heavens, no,” exclaimed Professor Smith, the eminent number theorist. “My children are playing with friends from three other families in the neighborhood, although our family happens to be largest. The Browns have a smaller number of children, the Greens have a still smaller number, and the Blacks the smallest of all.”

“How many children are there altogether?” asked Jones.

“Let me put it this way,” said Smith. “There are fewer than 18 children, and the product of the numbers in the four families happens to be my house number which you saw when you arrived.”

Jones took a notebook and pencil from his pocket and started scribbling. A moment later he looked up and said, “I need more information. Is there more than one child in the Black family?”

As soon as Smith replied, Jones smiled and correctly stated the number of children in each family.

Knowing the house number and whether the Blacks had more than one child, Jones found the problem trivial. it is a remarkable fact, however, that the number of children in each family can be determined solely on the basis of the information given above! What are these numbers?

(From Martin Gardner)

8 Hiker

A hiker left her camp, hiked 10 miles south, then 10 miles east, ate lunch, then hiked 10 miles north and was back at her camp. Where was her camp?

9 The Attic Lamp Switch

A downstairs panel contains three on-off switches, one of which controls the lamp in the attic—but which one? Your mission is to do something with the switches, then determine after *one* trip to the attic which switch is connected to the attic lamp.

10 Number Guessing Cards

Here are two versions of the Number Guessing Cards. One person chooses a number that appears on at least one card, then indicates on which cards the chosen number appears. The other person then must as quickly as possible determine which number was chosen.

Small Set

CARD 1	**CARD 2**	**CARD 3**
1	2	4
3	3	5
5	6	6
7	7	7

Large Set

CARD 1	CARD 2	CARD 3	CARD 4	CARD 5
1	2	4	8	16
3	3	5	9	17
5	6	6	10	18
7	7	7	11	19
9	10	12	12	20
11	11	13	13	21
13	14	14	14	22
15	15	15	15	23
17	18	20	24	24
19	19	21	25	25
21	22	22	26	26
23	23	23	27	27
25	26	28	28	28
27	27	29	29	29
29	30	30	30	30
31	31	31	31	31

11 Handshaking Couples

My wife and I recently attended a party at which there were four other married couples. Various handshakes took place. No one shook hands with himself (or herself) or with his (or her) spouse, and no one shook hands with the same person more than once. After all the handshakes were over, I asked each other person, including my wife, how many hands he (or she) had shaken. To my surprise each gave a different answer. How many hands did my wife shake?

See also Zeitz, Example 1.1.4.

12 Thinking about Thinking—Cards

Two students, A and B, are chosen from a math class of highly logical individuals. They are each given one positive integer. Each knows his/her own number, and is trying to determine the other's number. They are informed that their numbers are consecutive. In each of the following scenarios, what can you deduce about the two numbers?

1. First Scenario

A: I know your number.

B: I know your number.

2. Second Scenario

   A: I don't know your number.

   B: I know your number.

   A: I know your number.

3. Third Scenario

   A: I don't know your number.

   B: I don't know your number.

   A: I know your number.

   B: I know your number.

4. Fourth Scenario

   A: I don't know your number.

   B: I don't know your number.

   A: I don't know your number.

   B: I don't know your number.

   A: I know your number.

   B: I know your number.

13 Double Chess

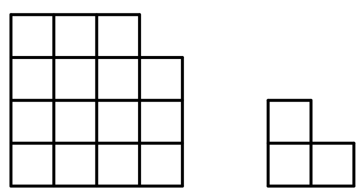
How can you simultaneously play one game of chess with one chessmaster, and another game of chess with another chessmaster, and be able to lose at most one of the games.

14 Animal Colors

Write down a three digit number x in which no digit is zero and the first digit is larger than the last digit. Write this number backwards and subtract the reversal from the original number x. Call the result y. (If y is a two-digit number, put a zero in front of it to regard it as having three digits.) Now reverse y and add the reversal to y. Call the result z. Add up the digits of z. Subtract 14. Use the result to obtain a single letter of the alphabet: 1 is A, 2 is B, etc. Think of a country whose name begins with this letter. Look at the second letter of this country's name and think of an animal that begins with that letter. Think of a color that this animal is likely to have.

15 Tiling with L's

For n a positive integer consider an array of $2^n \times 2^n$ squares, with the upper right-hand square removed. Prove that this array can be tiled by "L's" consisting of three squares. In the figure below we show the array $n = 2$, and one "L".



16 Newspaper Sheet

A single unfolded sheet of newspaper is placed on the floor. Two individuals simultaneously stand on this sheet but are unable to touch each other. What is the simplest explanation?

17 Towers of Hanoi

The Towers of Hanoi puzzle consists of three posts and n disks of different sizes that are initially stacked on one of the posts, from largest to smallest with the largest disk on the bottom. The goal is to move this stack of disks to a different post by a sequence of moves. Each move consists of moving the disk from top of the pile on any post to the top of the pile on any other post, as long as a larger disk is never placed directly on a smaller one. Try to solve this puzzle with $n <= 7$. You can play this game at the National Library of Virtual Manipulatives, `http://nlvm.usu.edu`.

1. What is the minimum number of moves needed to solve the puzzle?
2. What is an algorithm (recursive or nonrecursive) that can be used to solve the puzzle?

18 Thinking about Thinking—Hats

Three students — Alfred, Beth, and Carla — are blindfolded and told that either a red or a green hat will be placed on each of them. After this is done, the blindfolds are removed; the students are asked to raise a hand if they see a red hat, and to leave the room as soon as they are sure of the color of their own hat. All three hats happen to be red, so all three students raise a hand. Several minutes go by until Carla, who is more astute than the others, leaves the room. How did she deduce the color of her hat?

19 Six Triangles

What is the simplest way to form four equilateral triangles, each side being one foot in length, from six one foot rulers?

20 East and West

What is the biggest city in the U.S. east of Reno, Nevada and west of Denver, Colorado?

21 Two Pebbles

You have been falsely accused of a terrible crime, and your fate (execution or innocence) is to be determined by a random drawing. In a sack the king will place one small white pebble and one small black pebble. You will draw one pebble randomly from the sack. If it is white, you will be declared innocent and freed, but if it is black, you will be executed. Unfortunately for you, the king hates you, and has placed two black pebbles in the sack instead of one of each color. You learn of this in time, but dare not accuse the king of cheating—that would also result in your immediate execution. What should you do?

22 Binomial Coefficients

For nonnegative integer n consider the expansion of $(x+y)^n$,

$$(x+y)^n = \sum_{k=0}^{n} c_{n,k} x^k y^{n-k}.$$

Prove that

$$c_{n,k} = \frac{n!}{k!(n-k)!}, \ \ k = 0, \dots, n.$$

23 Permutations

Suppose n is a nonnegative integer. How many ways are there to arrange n different books on a shelf? Argue this from scratch, without relying on a previously encountered formula.

24 Choosing and Permuting

Suppose n and k are nonnegative integers and you have a box containing n different books. How many ways are there of choosing and arranging k of these books on a shelf? Argue this from scratch, without relying on a previously encountered formula.

25 Choosing

Suppose n and k are nonnegative integers and you have a box containing n different books. How many ways are there of choosing k of these books to put in your backpack in no particular arrangement? Argue this from scratch, without relying on a previously encountered formula.

26 Binomial Coefficients, Revisited

Think again about Problem 22, Binomial Coefficients. Use insight from Problem 25, Choosing, to obtain a different proof of the formula for binomial coefficients.

27 Friday

A rancher rides into town on Friday. Two days later he leaves on Friday. How can this be?

28 The Incident in the Bar

A man walks into a bar and asks for a glass of water. The bartender takes out a pistol and shoots at the ceiling. The man thanks the bartender and leaves. What is going on here?

29 What's My Rule?

One person creates a rule for sequences of three numbers, and provides an example. The other person tries to guess the rule by providing test cases of other sequences of three numbers.

30 Creating Formulas

1. Write down three real numbers a, b, c. Quickly write down any function $y = f(x)$ such that $f(a) = f(b) = f(c) = 0$.

2. Now quickly write down a function such that $f(a) = f(b) = f(c) = 0$ but $f(x) \neq 0$ for all $x \notin \{a, b, c\}$.

3. Extend the previous problem to more than three numbers.

4. Write down six real numbers a, b, c, p, q, r. Write down a function $y = f(x)$ such that $f(a) = p$, $f(b) = q$, and $f(c) = r$.

5. Generalize the previous problem and its solution in several ways.

6. Below is a table of values for a certain function $y = f(x)$. Construct a plausible closed formula for $f(x)$. Is the answer unique?

x	0	1	2	3	4	5
$f(x)$	1	2	4	7	11	16

7. Let $f(n)$ be the sum $1^2 + 2^2 + 3^2 + \cdots + n^2$. Create a table, and then construct a plausible closed formula for $f(n)$, $n \geq 1$. Now prove that your formula is correct.

8. Let $f(n)$ be the sum $1^3 + 2^3 + 3^3 + \cdots + n^3$. Create a table, and then construct a plausible closed formula for $f(n)$, $n \geq 1$. Now prove that your formula is correct.

31 Sum of Two Primes

Find all the ways to write the number 543 as the sum of two primes.

32 27 Card Trick

This trick is described by Eric Shrader on Cut-the-Knot,
www.cut-the-knot.org/arithmetic/rapid/CardTrick.shtml.

"My grandfather taught me this simple trick when I was young:

"Deal out 27 cards face up into a grid of 9 rows and 3 columns. Do this by dealing out 3 cards horizontally in a row, then 3 more cards just below the first 3, then 3 more, etc., until you have 9 rows. (It's best to overlap cards in a column so that the columns aren't so long; just make sure that the values of all cards are visible.) Discard the remaining cards. Only these 27 will be used to play.

"Ask a spectator to mentally pick a card and remember it. Ask him to tell you only which of the 3 columns it is in.

"Collect the 27 cards into a deck. Gather them vertically such that the column containing the spectator's card is second. Pick up cards from the top of the column to the bottom, keeping them in the same order. For example, if the spectator tells you column 1, then first gather column 2 or 3, then column 1, and then the remaining column. When you're done, the top card of the deck should now be the top card of the column you collected first, followed by the rest of that column in order. Then the 10th card of the deck will be the card at the top of the column containing the spectator's card, etc. If you overlapped the cards as suggested, gathering and keeping them in the correct order is easy.

"Now deal the cards again. Deal exactly as described above—horizontally across the rows first.

"Again, ask him to tell you only which column contains the card.

"Pick up the columns vertically—exactly as above—making sure that the column containing the card is picked up second.

"Finally deal them out a third time in exactly the same way, ask which column contains the card, and gather them up in the same manner.

"The spectator's card will now be the 14th one in the deck. To add drama, I usually deal out the cards one at a time face down. I don't make it obvious that I'm counting and I don't look at any of the cards. Instead, I'll hesitate over certain cards, pretending to get a "vibe" from them, and then I'll finally settle on the right one.

"This seemed like pure magic when I first learned it. Only when I got older did I realize that it's actually simple math. Do you see how it works?"

33 8 Card Trick

This is a variant of the previous trick. This time use 8 cards (though it can be modified for more cards) and two columns. The magician asks one spectator to name (aloud) a whole number N from 0 to 7, and a second spectator to select a card. Cards are dealt and collected three times as before, but now, at the very end after the final gathering, the magician is able to count off precisely N cards, revealing the chosen card as the next one. By what method does the magician stack the columns to force the chosen card to the desired position?

34 27 Card Trick Variants

1. This is a variant of the previous trick. This time use 9 cards and three columns. The magician asks one spectator to name (aloud) a whole number N from 0 to 8, and a second spectator to select a card. Cards are dealt and collected two times, but now, at the very end after the final gathering, the magician is able to count off precisely N cards, revealing the chosen card as the next one. By what method does the magician stack the columns to force the chosen card to the desired position?

2. This time use 27 cards and three columns. The magician asks one spectator to name (aloud) a whole number N from 0 to 26, and a second spectator to select a card. Cards are dealt and collected three times, but now, at the very end after the final gathering, the magician is able to count off precisely N cards, revealing the chosen card as the next one. By what method does the magician stack the columns to force the chosen card to the desired position?

35 Counterfeit Coins

1. Suppose you have a collection of nine coins, eight of which are of equal weight, and the ninth which is counterfeit and a little bit heavier. How can you use a two-pan balance scale to perform weighings of various piles of the coins to determine which coin is counterfeit? How many weighings are required to *guarantee* finding the counterfeit coin?

2. Generalize the above problem and its solution to the case when the number of coins is a power of 3.

3. Now suppose you have twelve coins, among which is one counterfeit coin, which may be either a little bit lighter or a little bit heavier. How many weighings will you need

to guarantee finding the counterfeit coin?

See the National Library of Virtual Manipulatives, `http://nlvm.usu.edu`, Coin Problem.

36 How Many Questions?

You are going to play a game with a friend. Your friend is going to think of a whole number in the range 0 to 127, inclusive. You are going to ask "yes" or "no" questions to determine the number. What is the fewest number of questions that you can ask to eventually determine the number?

37 Weird Multiplication I

Why does the following method work?

Suppose you want to calculate 18×14. Write these two numbers at the top of two columns. In the first column, start with 18, divide it by two, and repeat, ignoring any remainders, until you reach the number 1. In the second column, start with 14, double it, and repeat, until you have a column of equal length. Draw a line through any number in the first column that is even, and extend the line to cross out the neighboring number in the second column. Add up the remaining numbers in the second column. This will be the product, and this procedure works for any pair of numbers. Two examples:

~~18~~	~~14~~
9	28
~~4~~	~~56~~
~~2~~	~~112~~
1	224
	252

13	25
~~6~~	~~50~~
3	100
1	200
	325

38 Weird Multiplication II

Why does the following method work?

Number the fingers on each hand 5, 6, 7, 8, and 9, starting with your pinkies and ending with your thumbs. Hold your hands out in front of you with the palms facing you, thumbs up, and pinkies down.

To multiply two numbers on the above list together, touch the corresponding fingers together. For example, to calculate 8×7, touch the index finger of your left hand to the middle finger of your right hand.

At and above the index finger on your left hand there are 2 fingers. At and above the middle finger on your right hand there are 3 fingers. Multiply these two numbers together: $2 \times 3 = 6$. The remaining fingers on both hands combined each count 10, and there are 5 of them, for a product of 50. Add 6 and 50 to get the answer: $8 \times 7 = 56$.

Confused? Let's try another one: 6×8. Touch the ring finger of your left hand to the index finger of your right hand. Multiplying the number of fingers at and above the touching fingers: $4 \times 2 = 8$. Counting fingers below the touching fingers as tens: 40. Add 8 and 40 to get the answer: 48.

39 The Average Traveler

1. There are two towns, conveniently called "A" and "B" joined by a road. A business woman must drive from A to B, conduct some business (what else?) and return to A. She wants her total round trip speed to average 60 miles per hour, but she encountered heavy traffic going to B and averaged only 30 miles per hour on the first half of the trip. What should her average speed be on the return trip to meet her goal?

2. Now answer the question if on the first half of the trip she averaged 50 miles per hour.

3. Now answer the question if on the first half of the trip she averaged 31 miles per hour.

40 50 Words

You have 20 seconds to provide 50 words that each do not contain the letter "A".

41 Two-Cube Calendar

I once saw in a store an unusual desk calendar consisting of two cubes that could be arranged side-by-side in a frame. The date was indicated simply by arranging the two cubes so that their front faces gave the date. The face of each cube bore a single digit, 0 through 9, and one could arrange the cubes so their front faces indicated any date from 01, 02, 03, ..., to 31. How should the faces of the two cubes be labeled so that all the dates can be displayed? (I confess there is a little "twist" to the solution.)

42 Venn Diagrams

What does a Venn diagram with three circles have to do with writing the numbers from 0 to 7 in base two?

43 Watching TV

In a suburban home live Abner, his wife Beryl and their three children, Cleo, Dale and Ellsworth. The time is 8 p.m. on a winter evening.

1. If Abner is watching television, so is his wife.
2. Either Dale or Ellsworth, or both of them, are watching television.
3. Either Beryl or Cleo, but not both, is watching television.
4. Dale and Cleo are either both watching or both not watching television.
5. If Ellsworth is watching television, then Abner and Dale are also watching.

Who is watching television and who is not?

44 Logical Implications in Algebraic Reasoning

1. Solve $x^2 = 25$.
2. Solve $x^2 < 4$.
3. Solve $x^2 > 5$.
4. Solve $|x - 3| > 5$.
5. Solve $|x + 1| \leq 6$.
6. Solve $|x + 1| + |x - 1| = 2$.
7. Solve $\frac{1}{x^2-1} = \frac{1}{3x+3}$.
8. Solve $\frac{x^2}{x-1} = \frac{2-x}{x-1}$.
9. Solve $\frac{1}{\sqrt{x^2-1}} \geq \frac{1}{\sqrt{3x+3}}$.

10. Solve $x(2x+3) = x(x-5)$.

11. Solve $\frac{1}{\frac{1}{x}} = x$.

12. Solve $\sqrt{x^2 - 5x + 5} = \sqrt{x-3}$.

45 Outdoor Barbecue

Tom, John, Fred, and Bill are friends whose occupations are (in no particular order) nurse, secretary, teacher, and pilot. They attended a picnic recently, and each one brought his favorite meat (hamburger, chicken, steak, and hot dogs) to barbecue. From the clues below, determine each man's occupation and favorite meat.

1. Tom is neither the nurse nor the teacher.
2. Fred and the pilot play in a jazz band together.
3. The burger lover and the teacher are not musically inclined.
4. Tom brought hot dogs.
5. Bill sat next to the burger fan and across from the steak lover.
6. The secretary does not play an instrument or sing.

46 Smith, Jones, and Robinson

The following seven facts are given:

1. Smith, Jones and Robinson are the engineer, brakeman and fireman on a train, but not necessarily in that order. Riding the train are three passengers with the same three surnames, to be identified in the following premises by a "Mr." before their names.
2. Mr. Robinson lives in Los Angeles.
3. The brakeman lives in Omaha.
4. Mr. Jones long ago forgot all the algebra he learned in high school.
5. The passenger whose name is the same as the brakeman's lives in Chicago.

6. The brakeman and one of the passengers, a distinguished mathematical physicist, attend the same church.
7. Smith beat the fireman at billiards.

Who is the engineer?

.

47 Five Houses

There are five houses, each a different color and inhabited by men of different nationalities with different pets, drinks, and favorite foods.

1. The Englishman lives in the red house.
2. The Spaniard owns the dog.
3. Coffee is drunk in the green house.
4. The Ukrainian drinks tea.
5. The green house is immediately to your right of the white house.
6. The pizza eater owns snails.
7. Sushi is eaten in the yellow house.
8. Milk is drunk in the middle house.
9. The Norwegian lives in the first house on the left.
10. The man who eats burritos lives next to the man with the fox.
11. Sushi is eaten next to the house with a horse.
12. The steak eater drinks orange juice.
13. The Japanese eats jambalaya.
14. The Norwegian lives next to the blue house.

Who drinks water and who owns the zebra?

48 Socks

In a bureau drawer there are 60 socks, all identical except for their color: 10 pairs are red, 10 are blue, and 10 are green. The socks are all mixed up in the drawer, and the room the bureau is in is totally dark. What is the smallest number of socks you must remove to be sure you have at least one matching pair?

49 Repeating Decimals

Prove that when a fraction a/b is expressed in decimal form, the resulting number will be either a terminating decimal or one that repeats with a period no longer than b.

50 Five Points in a Square

Prove that if five points are placed anywhere on or in a square of side length 1, at least two points will be no farther apart than $\sqrt{2}/2$.

51 Five Points in a Triangle

Prove that if five points are placed anywhere on or in an equilateral triangle of side length 1, at least two points will be no farther apart than 1/2.

52 Faces in a Polyhedron

Try counting the edges around the faces of a polyhedron. You will find that there are always two faces somewhere bounded by the same number of edges. Why?

53 Same Sums

Prove that no matter how a set S of 10 positive integers smaller than 100 is chosen there will always be two disjoint subsets of S that have the same sum. For example, in the set 3, 9, 14, 21, 26, 35, 42, 59, 63, 76 there are the selections 14, 63, and 35, 42, both of which add to 77; similarly, the selection 3, 9, 14 adds up to 26, a number that is a member of the set.

54 Consecutive Pills

A physician testing a new medication instructs a test patient to take 48 pills over a 30-day period. The patient is at liberty to distribute the pills however he likes over this period as long as he takes at least one pill a day and finishes all 48 pills by the end of the 30 days. Prove that no matter how the patient decides to arrange things, there will be some stretch of consecutive days in which the total number of pills taken is 11.

55 Forcing Divisors

Suppose some set of 101 numbers $a_1, \ldots, a_{100}$ is chosen from the numbers $1, 2, 3, \ldots, 100$. Prove that it is impossible to choose such a set without taking two numbers for which one divides the other evenly; that is, with no remainder.

56 Subsequences

1. Take the numbers from $1, 2, 3, \ldots, n^2 + 1$ and arrange them in a sequence in any order. Prove that when the arrangement is scanned from left to right, it must contain either an increasing subsequence of length (at least) $n + 1$ or a decreasing subsequence of length (at least) $n+1$. For example, when $n = 3$, the arrangement 6,5,9,3,7,1,2,8,4,10 includes the decreasing subsequence 6,5,3,1. (As this example demonstrates, a subsequence need not consist of consecutive elements of the arrangement.)

2. Prove that in any sequence of $mn+1$ distinct real numbers there must be either an increasing subsequence of length (at least) $m + 1$ or a decreasing subsequence of length (at least) $n + 1$.

57 Lattice Points on Line Segments

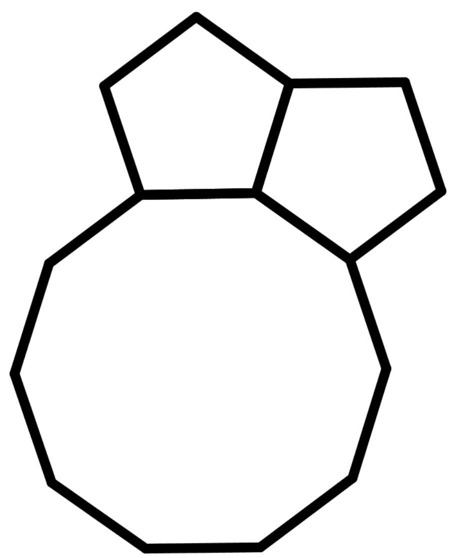
A lattice point is a point in a coordinate plane for which both coordinates are integers. Prove that no matter what five lattice points might be chosen in the plane at least one of the segments that joins two of the chosen points must pass through some lattice point in the plane.

58 Mutual Acquaintances

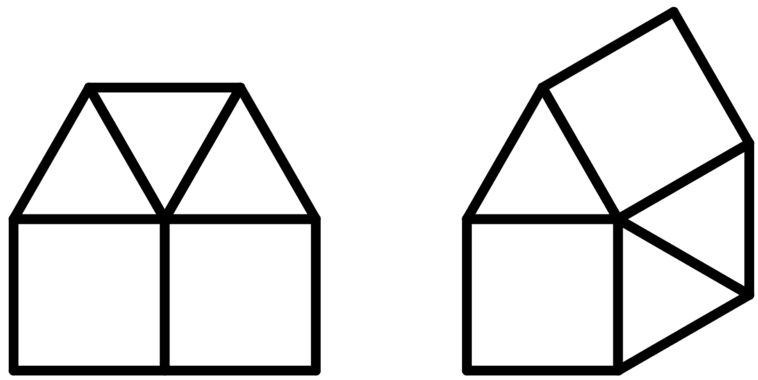
Show that for every group of six individuals there are always at least three people that mutually know each other, or three people that mutually do not know each other. (Assume that if person A knows person B, then also person B knows person A.)

59 Planar Clusters

Consider the problem of *planar clusters* of regular polygons fitting perfectly together around a single vertex. One regular decagon and two regular pentagons can fit together perfectly in the plane surrounding a common vertex (since the interior angle of a regular decagon measures 144 degrees, and the interior angle of a regular pentagon measures 108 degrees). Let's call this a (10,5,5) cluster (or a (5,10,5) cluster, or a (5,5,10) cluster).



Similarly, two squares and three equilateral triangles can fit together perfectly surrounding a common vertex. There are essentially two different ways to do this: (3,3,3,4,4) (where the squares are adjacent) and (3,3,4,3,4) (where the squares are not adjacent), and we will regard these as two *different* clusters.

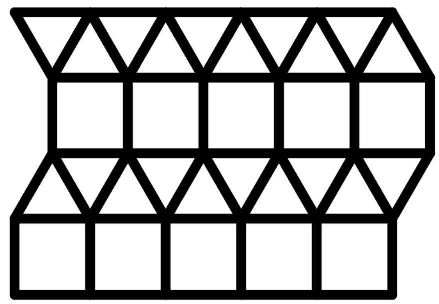


Note that we could have called this last cluster (4,3,3,4,3) as well—it still refers to the same cluster. However, (3,3,3,4,4) and (3,3,4,3,4) are *not* the same.

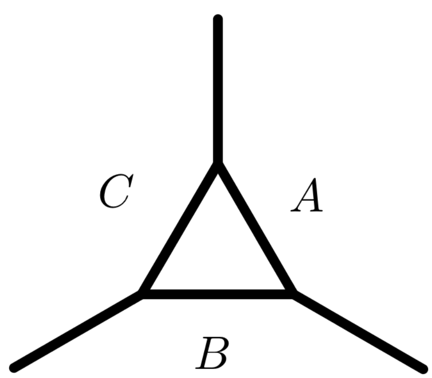
1. What is the minimum number of regular polygons that can form a planar cluster?
2. What is the maximum number of regular polygons that can form a planar cluster?
3. You have now seen at least three planar clusters. Determine all possible planar clusters that can be formed by fitting together a combination of three regular polygons in the plane surrounding a common vertex. Be *systematic* in some fashion, so that you can be certain you have found all of them, and explain clearly how you know this.
4. Explain why the previous problem is equivalent to finding combinations of positive integers $a, b, c \geq 3$ such that
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}.$$
5. Determine all possible planar clusters that can be formed by fitting together a combination of more than three regular polygons in the plane surrounding a common vertex.

60 Regular and Semiregular Tilings

Some of the clusters in the previous problem can be extended to tile the plane so that at every vertex exactly the same cluster appears—the same sequence of polygons, in either clockwise or counterclockwise order. For example, if you extend the (4,4,4,4) cluster, you get the familiar tiling of the plane with squares, with four squares meeting at each vertex, and if you extend the (3,3,3,4,4) cluster, you get the following tiling of the plane:



1. Of the clusters you have found, determine which ones can be extended to tilings of the plane. Make a precise drawing of each of the tilings that you find.
2. Prove that the (3,10,15) cluster cannot be extended to tile the plane. You may wish to consider the following diagram. If polygon A is a 15-gon, what must polygon B be? How about polygon C?



3. Choose another planar cluster with three polygons meeting at the common vertex that cannot be extended to a tiling, and develop a proof that the extension is not possible.
4. Choose a planar cluster with four polygons meeting at the common vertex that cannot be extended to a tiling, and develop a proof that the extension is not possible.