1 Axiomatic Systems

1.1 Features of Axiomatic Systems

One motivation for developing axiomatic systems is to determine precisely which properties of certain objects can be deduced from which other properties. The goal is to choose a certain fundamental set of properties (the *axioms*) from which the other properties of the objects can be deduced (e.g., as *theorems*). Apart from the properties given in the axioms, the objects are regarded as *undefined*.

As a powerful consequence, once you have shown that any particular collection of objects satisfies the axioms *however unintuitive or at variance with your preconceived notions these objects may be*, without any additional effort you may immediately conclude that all the theorems must also be true for these objects.

We want to choose our axioms wisely. We do not want them to lead to contradictions; i.e., we want the axioms to be *consistent*. We also strive for economy and want to avoid redundancy—not assuming any axiom that can be proved from the others; i.e., we want the axiomatic system to be *independent*. Finally, we may wish to insist that we be able to prove or disprove any statement about our objects from the axioms alone. If this is the case, we say that the axiomatic system is *complete*.

We can verify that an axiomatic system is consistent by finding a *model* for the axioms—a choice of objects that satisfy the axioms.

We can verify that a specified axiom is independent of the others by finding two models—one for which all of the axioms hold, and another for which the specified axiom is false but the other axioms are true.

We can verify that an axiomatic system is complete by showing that there is essentially only one model for it (all models are *isomorphic*); i.e., that the system is *categorical*.

Reference: Kay, Section 2.2.

1.2 Examples

Let's look at three examples of axiomatic systems for a collection of committees selected from a set of people. In each case, determine whether the axiomatic system is consistent or inconsistent. If it is consistent, determine whether the system is independent or redundant, complete or incomplete.

1. (a) There is a finite number of people.
   (b) Each committee consists of exactly two people.
   (c) Exactly one person is on an odd number of committees.

2. (a) There is a finite number of people.
   (b) Each committee consists of exactly two people.
   (c) No person serves on more than two committees.
   (d) The number of people who serve on exactly one committee is even.

3. (a) Each committee consists exactly two people.
   (b) There are exactly six committees.
   (c) Each person serves on exactly three committees.

1.3 Kirkman's Schoolgirl Problem

Consider the following axiomatic system for points and lines, where lines are certain subsets of points, but otherwise points and lines are undefined.

- Given any two distinct points, there is exactly one line containing both of them.
- Given any two distinct lines, they intersect in a single point.
- There exist four points, no three of which are contained in a common line.
- The total number of points is finite.

1. Try to come up with some models for this axiomatic system.
2. Find a model containing exactly 15 points.
3. Solve the following famous puzzle proposed by T. P. Kirkman in 1847:

   > *A school-mistress is in the habit of taking her girls for a daily walk. The girls are fifteen in number, and are arranged in five rows of three each, so that each girl might have two companions. The problem is to dispose them so that for seven consecutive days no girl will walk with any of her school-fellows in any triplet more than once.*

1.4 Finite Projective Planes

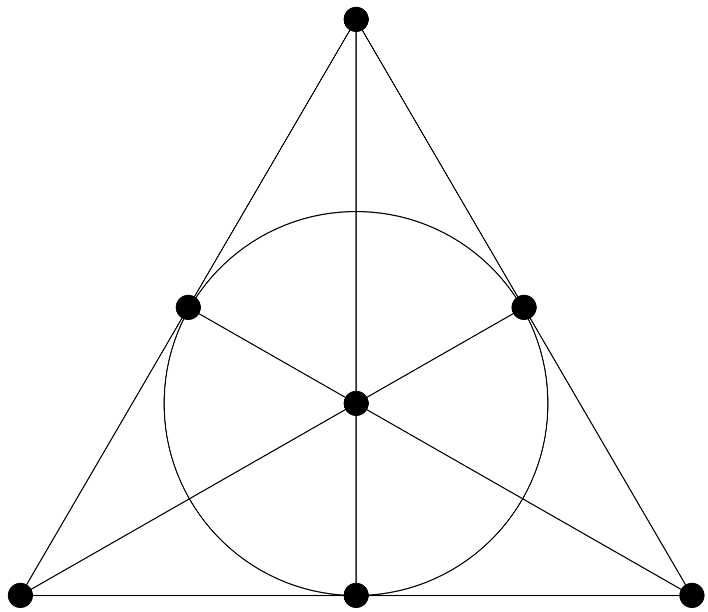
The axioms described in the previous section define structures called *finite projective planes*. I have a game called *Configurations* that is designed to introduce the players to the existence, construction, and properties of finite projective planes. When I checked in August 1997 the game was available from WFF 'N PROOF Learning Games Associates, 1490 South Boulevard, Ann Arbor, MI 48104, phone: (313) 665-2269, fax: (313) 764-9339, for a cost of $12.50.

Here are examples of some problems from this game:

1. In each box below write a number from 1 to 7, subject to the two rules: (1) The three numbers in each column must be different; (2) the same pair of numbers must not occur in two different columns.

	Col 1	Col 2	Col 3	Col 4	Col 5	Col 6	Col 7
Row 1							
Row 2							
Row 3							

2. Use the solution to the above problem to label the seven points of the following diagram with the numbers 1 through 7 so that the columns of the above problem correspond to the triples of points in the diagram below that lie on a common line or circle.



This is called the *Fano plane*.