4 Plane Separation

Here is a summary of the material in Section 2.6 of Kay.

Definition: A set K of points is *convex* if, for all distinct points $A, B \in K$, $\overline{AB} \subseteq K$.

Axiom H-1: Plane Separation Postulate: Let ℓ be any line in any plane P. The set of points of P not on ℓ consists of the union of two subsets H_1, H_2 of P such that:

1. H_1 and H_2 are convex sets.
2. $H_1 \cap H_2 = \emptyset$.
3. If $A \in H_1$ and $B \in H_2$ then $\ell \cap \overline{AB} \neq \emptyset$.

Definition: In the above situation H_1 and H_2 are *half-planes*, are called the two *sides* of ℓ, and we write $H_1 = H(A, \ell)$ and $H_2 = H(B, \ell)$.

Exercise 4.0.2 Determine whether or not Axiom H-1 holds in each of the geometric worlds $\mathbf{E}^2$, $\mathbf{S}^2$, $\mathbf{U}^2$, and $\mathbf{H}^2$.

Lemma: If $B \in \ell$, $A \in H_1$ (one of the two sides of ℓ), and A-B-C, then C is in the opposite side of ℓ.

Lemma: Half-planes are nonempty.

Definition:

1. $\overset{\circ\!-\!-\!\circ}{AB} = \{X : A\text{-}X\text{-}B\}$, *open segment.* The book uses the notation $(\overline{AB})$.

2. $\overset{\circ\!\longrightarrow}{AB} = \{X : X = B,\ A\text{-}X\text{-}B,\ \text{or}\ A\text{-}B\text{-}X\}$, *open ray.* The book uses the notation $(AB]$.

3. $[H_1] = H_1 \cup \ell$, where $H_1 = H(P, \ell)$, *closed half-plane.*

Theorem 2.6.1: If one point of a segment or ray lies in a half-plane H_1 determined by some line ℓ, and the endpoint of the segment or ray itself lies on ℓ, then the entire open segment or open ray lies in H_1. (This is Theorem 1 of Kay, Section 2.6.)

Corollary: Let B and F lie on opposite sides of a line ℓ and let A and G be any two distinct points on ℓ. Then $\overline{GB} \cap \overrightarrow{AF} = \emptyset$.

Theorem 2.6.2 (Postulate of Pasch): Suppose A, B, and C are any three distinct noncollinear points in a plane, and ℓ is any line which also lies in that plane and passes through an interior point D of segment $\overline{AB}$ but not through A, B, nor C. Then ℓ meets either $\overline{AC}$ at some interior point E, or $\overline{BC}$ at some interior point F, the cases being mutually exclusive. (This is Theorem 2 of Kay, Section 2.6.)

Definition: $\text{int}\angle ABC = H(A, \overleftrightarrow{BC}) \cap H(C, \overleftrightarrow{BA})$. I.e., the *interior* of $\angle ABC$ is the set of all points X which simultaneously lie on the A-side of $\overleftrightarrow{BC}$ and on the C-side of $\overleftrightarrow{BA}$.

Theorem 2.6.3: If A and C lie on the sides of $\angle B$, then, except for endpoints, segment $\overline{AC} \subseteq \text{int}\angle B$. If $D \in \text{int}\angle B$, then, except for B, $\overrightarrow{BD} \subseteq \text{int}\angle B$. (This is Theorem 3 of Kay, Section 2.6.)