Exam #3 — Take Home

Due in class Friday, December 5

Instructions: You may ask me questions and refer to the course notes and material, and you may use calculators and GeoGeobra, but otherwise you may not consult any other source of information or assistance, either human or inhuman. It will be helpful to refer to the document "Transformation Matrices" to be posted shortly to the course website.

1. For each of the matrices below, identify whether it represents a translation, rotation, reflection, or glide reflection. If it is a translation, determine the translation vector (p, q). If it is a rotation, determine the center of rotation (p, q), and the sine and the cosine of the rotation angle δ. If it is a reflection, determine the equation of the line of reflection, $px + qy = r$. If it is a glide reflection, determine the equation of the line of reflection $px + qy = r$ as well as the value of t. Show your work.

   (a)

$$\begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 2 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   (b)

$$\begin{bmatrix} -\frac{7}{25} & -\frac{24}{25} & 2 \\ -\frac{24}{25} & \frac{7}{25} & 14 \\ 0 & 0 & 1 \end{bmatrix}$$

2. (a) Write down the 3×3 matrix M_1 for the reflection across the line $y = a$ and the 3×3 matrix M_2 for the reflection across the line $y = b$.

   (b) Using M_1 and M_2, determine the matrix M_3 to describe the net effect of first reflecting across the line $y = a$ and then reflecting across the line $y = b$.

   (c) Verify that M_3 is a translation matrix, determine the vector of translation, and confirm that the translation is in a direction perpendicular to the two lines with translation amount equal to twice the distance between the two lines.

3. (a) Write down the 3×3 matrix M_1 for the reflection across the line $y = 0$ and the 3×3 matrix M_2 for the reflection across the line $y = x$.

   (b) Using M_1 and M_2, determine the matrix M_3 to describe the net effect of first reflecting across the line $y = 0$ and then reflecting across the line $y = x$.

   (c) Verify that M_3 is a rotation matrix, determine the center and angle of rotation, and confirm that the rotation is about the intersection of the two lines, and by an angle twice that of the angle between the two lines.

4. (a) The conjugate of a complex number $z = x + iy$ is the complex number $\overline{z} = x - iy$. Consider the function $f(z) = \overline{z}$. Explain why this function is a reflection of the plane across the line $y = 0$.

   (b) The reflection across the line $y = x$ can be obtained the following way. First rotate by -45 degrees about the origin, second take the complex conjugate, and third rotate by 45 degrees about the origin. Use compositions of functions of complex variables (not 3×3 matrices) to derive the formula for the reflection across the line $y = x$.

   (c) Now write down the 3×3 matrix for reflection across the line $y = x$ and confirm that you get the same formula as in (4b).

5. Let $A = (0, 0)$, $B = (16, -2)$, and $C = (11, 27)$. Use constructions (not just guess and check) in GeoGebra to find the point P such that $AP = 5$, $BP = 13$, and $CP = 25$. Describe what constructions you used.

6. Consider the following four matrices.

$$M_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 & -1 & 5 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} -1 & 0 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Verify that $M_2 = M_1^2$, $M_3 = M_1^3$, and $M_0 = M_1^4$.

(b) Interpret these matrices as isometries and then use this to explain why the formulas in (6a) make sense geometrically.