2.2 Geometrical Worlds

Problem 2.2.1 Here are some geometrical "worlds." In each case we make certain choices on what we will call POINTS and LINES. (I capitalize these words as a reminder these may not appear to be our "familiar" points, lines and planes.) In each case you should begin thinking about what properties hold for our choice of POINTS and LINES. In particular,

1. Is it true or false that given any two different POINTS, there is exactly one LINE that contains both of them?
2. Is it true or false that for any given LINE and any given POINT not on that LINE, there is a unique LINE containing the given POINT that does not intersect the given LINE?

It would be helpful for experimentation to have some spherical surfaces to draw on, such as (very smooth) tennis balls, ping-pong balls, oranges or Lénárt spheres.

2.2.1 The Analytical Euclidean Plane: $\mathbf{E}^2$

POINTS: Ordered pairs (x, y) of real numbers; i.e., elements of $\mathbf{R}^2$.

LINES: Sets of points that satisfy an equation of the form $ax + by + c = 0$, where a, b and c are real numbers; and further a and b are not both zero.

2.2.2 The Rational Plane

POINTS: Ordered pairs (x, y) of rational numbers; i.e., elements of $\mathbf{Q}^2$.

LINES: Sets of points that satisfy an equation of the form $ax + by + c = 0$, where a, b and c are rational numbers; and further a and b are not both zero.

2.2.3 The Integer Plane

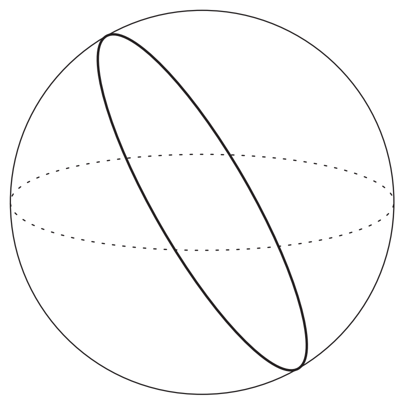
POINTS: Ordered pairs (x, y) of integers; i.e., elements of $\mathbf{Z}^2$.

LINES: Sets of points that satisfy an equation of the form $ax + by + c = 0$, where a, b and c are integers numbers; and further a and b are not both zero.

2.2.4 The Sphere: $\mathbf{S}^2$

POINTS: All points in $\mathbf{R}^3$ that lie on a sphere of radius 1 centered at the origin.

LINES: Great circles on the sphere (circles that divide the sphere into two equal hemispheres).



2.2.5 The Paired Sphere

(Note: This is not a standard name.)

POINTS: All pairs of points in $\mathbf{R}^3$ that lie on a sphere of radius 1 centered at the origin and are opposite each other.

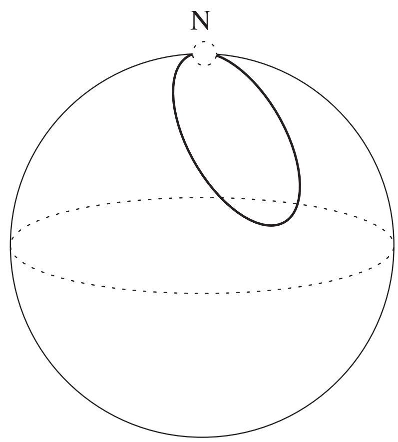
LINES: Great circles on the sphere.

2.2.6 The Punctured Sphere

(Note: This is not a standard name.)

POINTS: All points in $\mathbf{R}^3$ that lie on a sphere of radius 1 centered at the origin, with the exception of the point $N = (0, 0, 1)$ (the "North Pole"), which is excluded.

LINES: Circles on the sphere that pass through N, excluding the point N itself.

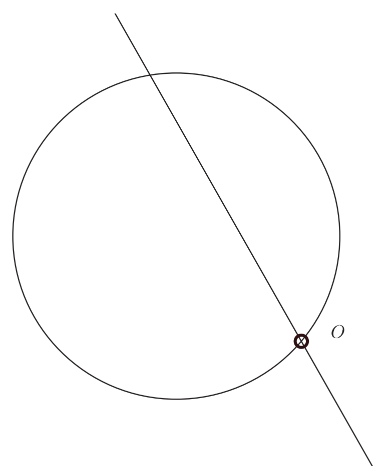


2.2.7 The "Inside-Out" Plane

(Note: This is not a standard name.)

POINTS: All points in $\mathbf{R}^2$, but not including the point $O = (0,0)$, together with an additional "artificial" POINT not in $\mathbf{R}^2$, which we will call Z.

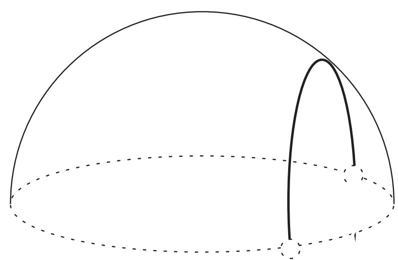
LINES: Circles passing through O but excluding the point O itself, together with ordinary lines passing through O (but not including O) together with the artificial POINT Z.



2.2.8 The Open Hemisphere

POINTS: All points in $\mathbf{R}^3$ that lie on the upper hemisphere of radius 1 centered at the origin and have strictly positive z-coordinate. (So the "equator" of points with z-coordinate equaling 0 is excluded.)

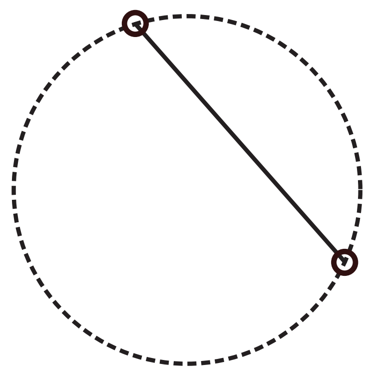
LINES: Semicircles (not including endpoints) on this hemisphere that are perpendicular to the "equator".



2.2.9 The Klein Disk

POINTS: All points in $\mathbf{R}^2$ that lie strictly in the interior of the circle of radius 1 centered at the origin.

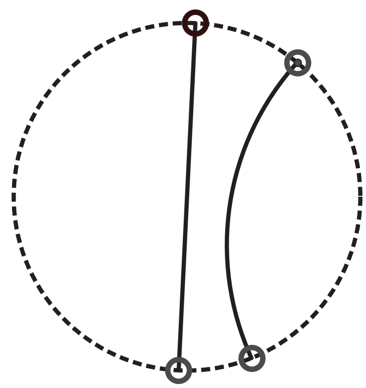
LINES: Chords of the circle, excluding endpoints.



2.2.10 The Poincaré Disk: $\mathbf{H}^2$

POINTS: All points in $\mathbf{R}^2$ that lie strictly in the interior of the circle C of radius 1 centered at the origin.

LINES: Points of $\mathbf{H}^2$ that lie on circles intersecting C in right angles, as well as diameters (excluding endpoints) of C.



2.2.11 The Upper Half Plane

POINTS: All points (x, y) in $\mathbf{R}^2$ for which $y > 0$.

LINES: Points of the upper half plane that lie on circles intersecting the x-axis in right angles, or that lie on vertical lines.

2.2.12 The Projective Plane: $\mathbf{P}^2$

POINTS: All ordinary lines in $\mathbf{R}^3$ that pass through the origin.

LINES: All ordinary planes in $\mathbf{R}^3$ that pass through the origin.

2.2.13 The Affine Plane: $\mathbf{A}^2$

POINTS: All ordinary nonhorizontal lines in $\mathbf{R}^3$ that pass through the origin.

LINES: All ordinary nonhorizontal planes in $\mathbf{R}^3$ that pass through the origin.

2.2.14 The First Vector Plane

(Note: This is not a standard name.)

POINTS: Ordered triples (x, y, z) of real numbers for which x, y, and z are not all zero. Also, (x_1, y_1, z_1) and (x_2, y_2, z_2) are regarded as equivalent (the same point) if one triple is a nonzero multiple of the other.

LINES: Ordered triples (a, b, c) of real numbers for which a, b, and c are not all zero. Also, (a_1, b_1, c_1) and (a_2, b_2, c_2) are regarded as equivalent (the same line) if one triple is a nonzero multiple of the other.

A POINT (x, y, z) is regarded as being on a LINE (a, b, c) if $ax + by + cz = 0$.

2.2.15 The Second Vector Plane

(Note: This is not a standard name.)

POINTS: Ordered triples (x, y, z) of real numbers for which z is nonzero. Also, (x_1, y_1, z_1) and (x_2, y_2, z_2) are regarded as equivalent (the same point) if one triple is a nonzero multiple of the other.

LINES: Ordered triples (a, b, c) of real numbers for which a and b are not both zero. Also, (a_1, b_1, c_1) and (a_2, b_2, c_2) are regarded as equivalent (the same line) if one triple is a nonzero multiple of the other.

A POINT (x, y, z) is regarded as being on a LINE (a, b, c) if $ax + by + cz = 0$.

2.2.16 Analytical Euclidean Space: $\mathbf{E}^3$

POINTS: Ordered triples (x, y, z) of real numbers.

LINES: Sets of points of the form...

PLANES: Sets of points that satisfy an equation of the form $ax + by + cz + d = 0$, where a, b, c and d are real numbers; and further a, b and c are not all zero.

2.2.17 Analytical Euclidean 4-Space: $\mathbf{E}^4$

POINTS:

LINES:

PLANES:

2.2.18 Analytical Euclidean n-Space: $\mathbf{E}^n$

Here, assume n is an integer greater than 3.

POINTS:

LINES:

PLANES: