2.11 Relationships Among the Geometrical Worlds

We saw that stereographic projection maps circles on the sphere to circles and lines in the plane, and vice versa. It turns out that stereographic projection preserves angles between curves. In particular, two curves intersect at right angles on the sphere if and only if their images intersect at right angles in the plane. Using these facts, we can establish that some of the models presented in Section 2.2 are isomorphic to each other.

Problem 2.11.1 Prove that the following models are all isomorphic to each other: The analytical Euclidean plane 2.2.1, the punctured sphere 2.2.6, the "inside-out" plane 2.2.7, the affine plane, 2.2.13, and the second vector plane 2.2.15.

Solution:

- *Punctured Sphere* $\longrightarrow$ *Analytical Euclidean Plane. Apply stereographic projection from the point* $(0,0,1)$ *to the plane given by* $z=0$*, and then omit the* z*-coordinate.*
- *Punctured Sphere* $\longrightarrow$ *Inside Out Plane. Apply stereographic projection from the point* $(0,0,-1)$ *to the plane given by* $z=0$ *(but also map the point* $(0,0,-1)$ *to* Z*), and then omit the* z*-coordinate.*
- *Affine Plane* $\longrightarrow$ *Analytical Euclidean Plane. Intersect non-horizontal lines and non-horizontal planes through the origin with the plane given by* $z=1$*, and then omit the* z*-coordinate.*
- *Second Vector Plane* $\longrightarrow$ *Affine Plane. Map the POINT* (x,y,z) *to the line through the origin spanned by that point, and the LINE* (a,b,c) *to the plane given by* $ax+by+cz=0$*.*

Problem 2.11.2 Prove that the following models are all isomorphic to each other: The paired sphere 2.2.5, the projective plane 2.2.12, and the first vector plane 2.2.14.

Solution:

- *Projective Plane* $\longrightarrow$ *Paired Sphere. Intersect lines through the origin and planes through the origin with the sphere.*

- *First Vector Plane $\longrightarrow$ Projective Plane. Map the POINT (x, y, z) to the line through the origin spanned by that point, and the LINE (a, b, c) to the plane given by $ax + by + cz = 0$.*

Problem 2.11.3 Prove that the following models are all isomorphic to each other: the open hemisphere 2.2.8, the Klein disk 2.2.9, the Poincaré disk 2.2.10, and the upper half plane 2.2.11.

Solution:

- *Open Hemisphere $\longrightarrow$ Klein Disk. Apply orthogonal projection onto the plane given by $z = 0$, and then omit the z-coordinate.*
- *Open Hemisphere $\longrightarrow$ Poincaré Disk. Apply stereographic projection from the point $(0, 0, -1)$ to the plane given by $z = 0$, and then omit the z-coordinate.*
- *Open Hemisphere $\longrightarrow$ Upper Half Plane. Apply stereographic projection from the point $(1, 0, 0)$ to the plane given by $x = 0$, and then omit the x-coordinate.*

Problem 2.11.4 Think about how one can regard the Analytical Euclidean Plane 2.2.1 as sitting within the Projective Plane 2.2.12. (This helps make sense of "points at infinity" and a "line at infinity".)

Solution: *Think about the map above from the Affine Plane to the Analytical Euclidean Plane, and then how the Affine Plane is contained in the Projective Plane.*