MA 341 — Homework #2

Soutions

Solve the four problems on the Committees handout. *Justify all of your answers!* To address consistency you need to either produce one model, or else you have to explain how the system leads to a contradiction. To address independence, for *each* particular axiom you need to either produce two models (one in which all axioms hold, and the other in which the particular axiom does not hold but the others do), or else you have to explain how the axiom can be deduced (proved) from the other axioms in the system. To address the categorical property you need to either explain why there is essentially only one model (the same except for changing names of people or committees), or else produce two essentially different (nonisomorphic) models (there is no way to make one model look like the other by changing names of people or committees).

Solutions:

1. (a) This system is consistent. One possible model:
   People: 1,2,3,4.
   Committees:
   1,2
   1,3
   1,4
   2,3
   We can check that there are exactly four people and four committees, that each committee is different and has exactly two people.

   (b) Analyzing Independence

      i. Axiom A is independent. We can find a model in which Axioms B, C, and D are true, and Axiom A is not true. One possible model:
      People: 1,2,3,4,5.
      Committees:
      1,2
      1,3
      1,4
      2,3

      ii. Axiom B is independent. We can find a model in which Axioms A, C, and D are true, and Axiom B is not true. One possible model:
      People: 1,2,3,4.
      Committees:
      1,2

iii. Axiom C is independent. We can find a model in which Axioms A, B, and D are true, and Axiom C is not true. One possible model:
People: 1,2,3,4.
Committees:
1,2,3
1,3
1,4
2,3

iv. Axiom D is independent. We can find a model in which Axioms A, B, and C are true, and Axiom D is not true. One possible model:
People: 1,2,3,4.
Committees:
1,2
1,2
1,2
1,2

(c) This model is not categoric. Here are two essentially different models:

Model #1	Model #2
People: 1,2,3,4	People: 1,2,3,4
Committees:	Committees:
1,2	1,2
1,3	2,3
1,4	3,4
2,3	1,4

In the first model, person 1 serves on three committees. But in the second model, nobody serves on three committees. So these two models cannot be made to look the same by renaming or renumbering the people. It may be helpful for you to make a diagram of each of these models—the difference is then perhaps more apparent.

2. (a) This system is consistent. One possible model:
People: 1,2,3,4.
Committees:
1,2
1,3
1,4

2,3
2,4
We can check that there are exactly four people and five committees, that each committee is different and has exactly two people.

(b) Analyzing Independence

i. Axiom A is independent. We can find a model in which Axioms B, C, and D are true, and Axiom A is not true. One possible model:
People: 1,2,3,4,5.
Committees:
1,2
1,3
1,4
2,3
2,4

ii. Axiom B is independent. We can find a model in which Axioms A, C, and D are true, and Axiom B is not true. One possible model:
People: 1,2,3,4.
Committees:
1,2

iii. Axiom C is independent. We can find a model in which Axioms A, B, and D are true, and Axiom C is not true. One possible model:
People: 1,2,3,4.
Committees:
1,2,3
1,3
1,4
2,3
2,4

iv. Axiom D is independent. We can find a model in which Axioms A, B, and C are true, and Axiom D is not true. One possible model:
People: 1,2,3,4.
Committees:
1,2
1,2
1,2
1,2
1,2

(c) This model is categoric. The only way to make a model with four people is to choose all but one of the six possible committees of two people. So by renaming (or renumbering) the people, you can always make the model look like the model in (2a).

3. (a) This system is consistent. One possible model:

   People: 1,2,3,4.

   Committees:
   1,2
   1,3
   1,4
   2,3
   2,4
   3,4

   We can check that there are exactly four people and six committees, that each committee is different and has exactly two people, and that each person serves on exactly three committees.

   (b) Analyzing independence.

      i. Axiom A is not independent. If the other axioms are true, then Axiom A must be true as well. Justification: Suppose there are n people. We know from earlier reasoning that n (the number of people) times 3 (the number of committees each person is on, by Axiom E) counts each committee twice (since each committee has exactly two people, by Axiom C). Thus $3n = 12$ (since there are six committees, by Axiom B). This implies $n = 4$. So there must be exactly four people.

      ii. Axiom B is not independent. If the other axioms are true, then Axiom B must be true as well. Justification: We know that 4 (the number of people, by Axiom A) times 3 (the number of committees each person is on, by Axiom E) counts each committee twice (since each committee has exactly two people, by Axiom C). Let m be the number of committees. Thus $3 \times 4 = 2m$. This implies $m = 6$. So there must be exactly six committees.

      iii. Axiom C is independent. We can find a model in which Axioms A, B, D, and E are true, and Axiom C is not true. One possible model:

      People: 1,2,3,4

      Committees:
      1
      1,2,3

1,4
2,3
2,4
3,4

iv. Axiom D is independent. We can find a model in which Axioms A, B, C, and E are true, and Axiom D is not true. One possible model:
People: 1,2,3,4
Committees:
1,2
1,2
1,2
3,4
3,4
3,4

v. Axiom E is not independent. If the other axioms are true, then Axiom E must be true as well. Justification:
If you have four people, there are only six different possible committees with exactly two people. Thus, there is only one way to satisfy Axioms A through D; namely, with these six committees. But then each person serves on three committees, forcing Axiom E to be true as well.

(c) This system is categoric. The previous paragraph shows that there is essentially only one model—only the names or labels of the four people might be different, but that is not an essential difference. Any two models must be isomorphic—they can be made to be identical by renaming the people.

4. This system is not consistent. If you have four people, there are only six different possible committees with exactly two people. Thus, if Axioms A, C, and D are true, then B cannot also be true.