MA 341 Homework #5
Solutions

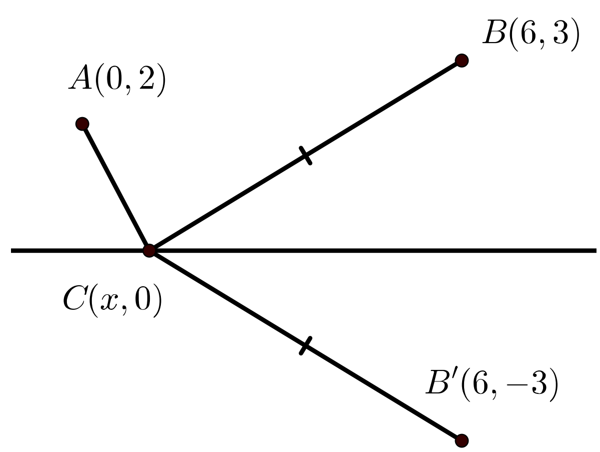
1. Consider the problem that we *already solved* using calculus.

> A camper finds herself near (but not at) the bank of a straight river. Describe how to construct the shortest path from her current location to her tent, given that she wishes first to stop by the river. If the river bank is represented by the line $y = 0$, her present location by the point $A = (0, 2)$, and her campsite by the point $B = (6, 3)$, what is the shortest route she can take? Provide justification. Make a good sketch. It may be helpful to use GeoGebra to experiment.

We figured out that we needed to find a particular point C on the river bank such that the angle that $\overline{AC}$ makes with the river matches the angle that $\overline{BC}$ makes with the river.

Now solve the problem a different, essentially geometric way, by inserting the point $B' = (6, -3)$ into the diagram and thinking about various line segments.

Solution. Reflect B across the line $y = 0$ to get the point $B' = (6, -3)$. Consider any point $C = (x, 0)$ on the line $y = 0$.



By reflection, $CB = CB'$, so $AC + CB = AC + CB'$. To minimize this latter sum, we must choose C to be collinear with A and B.

We can calculate the coordinates of C by our determinant formula for collinearity:

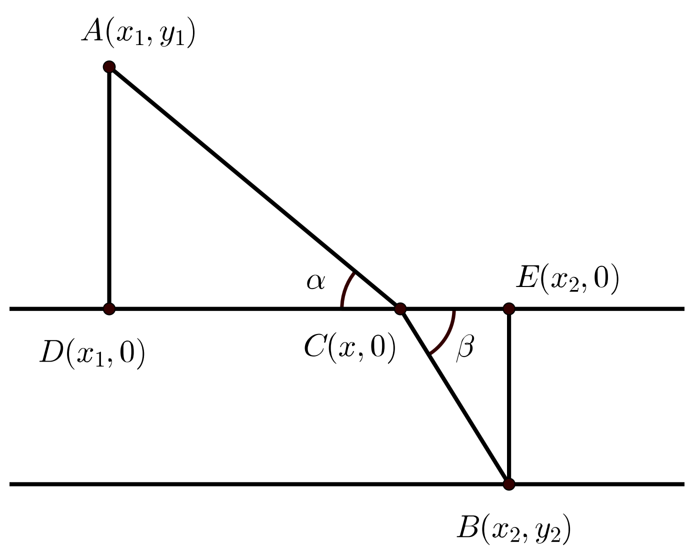
$$\det \begin{vmatrix} x & 0 & 6 \\ 0 & 2 & -3 \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

$$2x + 3x - 12 = 0.$$

So $x = 12/5$ and the camper should stop by the point $(2.4, 0)$.

2. A camper finds herself at a point $A(x_1, y_1)$ near (but not at) the bank of a straight river. Assume the closer bank of the river is given by the line $y = 0$. She can run at speed v and swim at speed w. She wants to get to a particular point $B(x_2, y_2)$ on the opposite bank of the river. So she runs to a point C on the near river bank and then swims from C to B. The water in the river is moving so slowly that during her swim you can neglect any movement downstream due to river flow. How can you determine the location of the point C that will minimize her total time?

   Solution. Let $C = (x, 0)$ be the point on the near river bank to which the camper will run, and swim from there to B.



To minimize time, we need to minimize the function

$$T(x) = \frac{AC}{v} + \frac{BC}{w}$$

with respect to x.

$$\begin{aligned}
T(x) &= \frac{((x-x_1)^2+(0-y_1)^2)^{1/2}}{v} + \frac{((x_2-x)^2+(y_2-0)^2)^{1/2}}{w} \\
T'(x) &= \frac{\frac{1}{2}((x-x_1)^2+y_1^2)^{-1/2}(2(x-x_1))}{v} + \frac{\frac{1}{2}((x_2-x)^2+y_2^2)^{-1/2}(-2(x_2-x))}{w} \\
T'(x) &= \frac{x-x_1}{v((x-x_1)^2+y_1^2)^{1/2}} - \frac{x_2-x}{w((x_2-x)^2+y_2^2)^{1/2}}.
\end{aligned}$$

As long as we don't have the special case when either $y_1 = 0$ or $y_2 = 0$ (and these cases are easy to handle) we can see that $T'(x)$ is always defined. We can also see from the diagram that it makes physical sense to restrict x to lie between x_1 and x_2. Set $T'(x) = 0$.

$$\frac{x-x_1}{v((x-x_1)^2+y_1^2)^{1/2}} - \frac{x_2-x}{w((x_2-x)^2+y_2^2)^{1/2}} = 0$$

$$\frac{x-x_1}{v((x-x_1)^2+y_1^2)^{1/2}} = \frac{x_2-x}{w((x_2-x)^2+y_2^2)^{1/2}}.$$

Looking at the diagram we can see that this is equivalent to

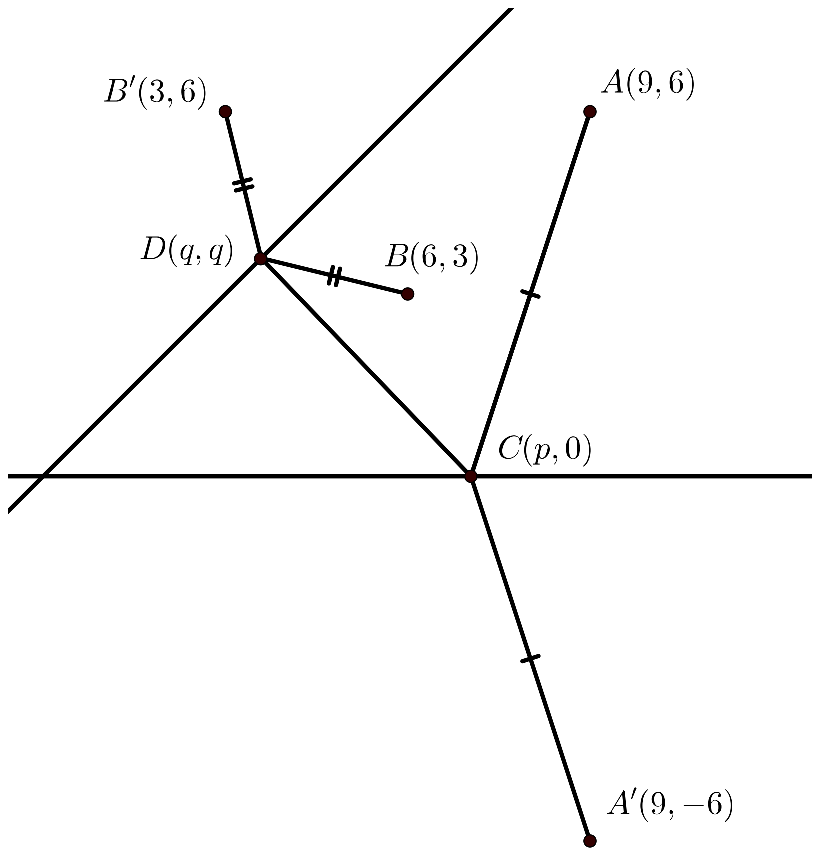
$$\begin{aligned}
\frac{DC}{vAC} &= \frac{EC}{wBC} \\
\frac{\cos\alpha}{v} &= \frac{\cos\beta}{w} \\
\frac{\cos\alpha}{\cos\beta} &= \frac{v}{w}.
\end{aligned}$$

So the point C should be chosen so that this condition is met.

3. A camper finds herself in the angle formed by the edge of a meadow and the bank of a river. Her tent is also in this angle. The bank of the river is given by line $y = 0$. The edge of the meadow is given by the line $y = x$. The camper is currently at the point $(9, 6)$, and the tent is at the point $(6, 3)$. Describe how to construct the shortest path

from her current location to her tent, given that she wishes first to stop by the river, and then after that stop by the meadow, on the way to her tent. What is the shortest path from her current location to the river to the meadow to the tent?

Solution.



Reflect the point $A(9,6)$ across the line $y=0$ to get the point $A'(9,-6)$, and reflect the point $B(6,3)$ across the line $y=x$ to get the point $B'(3,6)$. Consider any point $C(p,0)$ on the river bank and any point $D(q,q)$ at the edge of the meadow. By reflection, $AC = A'C$ and $BD = B'D$. So $AC + CD + DB = A'C + CD + B'D$. To minimize the latter sum, we need to select the points C and D so that the points A', C, D, B' are collinear. We can calculate the coordinates using our determinant formula for collinearity.

A', B', C are collinear, so

$$\det \begin{vmatrix} p & 9 & 3 \\ 0 & -6 & 6 \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

$$-6p + 54 - 6p + 18 = 0.$$

Thus $-12p + 72 = 0$ so $p = 6$ and the point C has coordinates $(6, 0)$.

A', B', D are collinear, so

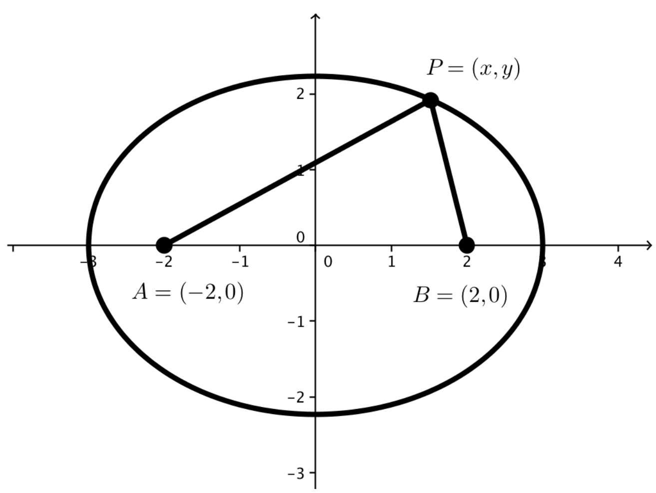
$$\det \begin{vmatrix} q & 9 & 3 \\ q & -6 & 6 \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

$$-6q + 54 + 3q - 6q - 9q + 18 = 0.$$

Thus $-18q + 72 = 0$ so $q = 4$ and the point C has coordinates $(4, 4)$.

4. Let $A = (-2, 0)$ and $B = (2, 0)$. Consider the set of all points $P = (x, y)$ such that the sum of the distances $PA + PB$ equals 6. Find an equation to describe this set of points, simplifying it as much as possible—in particular, figure out how to get rid of any square roots. Then use GeoGebra or a similar program to make a good sketch. What kind of shape do you get?

   Solution.



$$
\begin{aligned}
\sqrt{(x-2)^2+y^2}+\sqrt{(x+2)^2+y^2} &= 6 \\
\sqrt{x^2-4x+4+y^2} &= 6-\sqrt{x^2+4x+4+y^2} \\
x^2-4x+4+y^2 &= 36-12\sqrt{x^2+4x+4+y^2}+x^2+4x+4+y^2 \\
12\sqrt{x^2+4x+4+y^2} &= 36+8x \\
3\sqrt{x^2+4x+4+y^2} &= 9+2x \\
9(x^2+4x+4+y^2) &= 81+36x+4x^2 \\
5x^2+9y^2 &= 45 \\
\frac{x^2}{9}+\frac{y^2}{5} &= 1.
\end{aligned}
$$

This is an ellipse centered at the origin.

5. *It was the first time that Poole had seen a genuine horizon since he had come to Star City, and it was not quite as far away as he had expected.... He used to be good at mental arithmetic—a rare achievement even in his time, and probably much rarer now. The formula to give the horizon distance was a simple one: the square root of twice your height times the radius—the sort of thing you never forgot, even if you wanted to...*

—Arthur C. Clarke, *3001*, Ballantine Books, New York, 1997, page 71

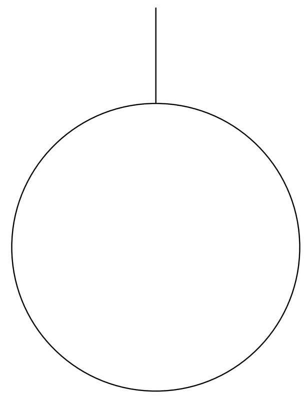
In the above passage, Frank Poole uses a formula to determine the distance to the horizon given his height above the ground.

(a) Use algebraic notation to express the formula Poole is using.

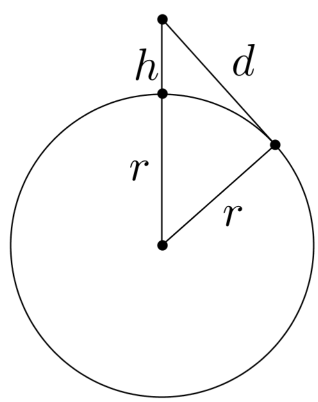
Solution. Let h denote the height above the ground, r denote the radius of the Earth and d denote the distance to the horizon. Then Poole's formula is

$$d=\sqrt{2hr}.$$

(b) Beginning with the diagram below, derive your own formula. You will need to add some more elements to the diagram.



Solution. Refer to the diagram below.



The radius and the tangent meet at a right angle. So by the Pythagorean Theorem, $(r+h)^2 = d^2 + r^2$, Solve for d to get $d = \sqrt{2hr + h^2}$.

(c) Compare your formula to Poole's; you will find that they do not match. How are they different?

Solution. There is an extra term of h^2 under the square root in the correct formula.

(d) When I was a boy it was possible to see the Atlantic Ocean from the peak of Mt. Washington in New Hampshire. This mountain is 6288 feet high. How far away is the horizon? Express your answer in miles. Assume that the radius of the

Earth is 4000 miles. Use both your formula and Poole's formula and comment on the results. Why does Poole's formula work so well, even though it is not correct?

Solution. Using $h = 6288/5280$ miles, to two significant figures Poole's formula gives $d = 98$ miles, and the correct formula also gives $d = 98$ miles. The reason Poole's formula gives a reasonable estimate is because h^2 is small in comparison to $2hr$ when h is small in comparison to r. But you can see that as h gets very large, Poole's formula can become quite inaccurate.