MA 341 Homework #8
Due Monday, November 24, in Class

1. Find all complex numbers z such that $z^3 = -8i$.

   Solution. Since $-8i$ has length 8 and angle 270 degrees, we need to find $z = rcis\theta$ such that $r^3 = 8$ and $3\theta = 270 + k360$ for $k = 0, 1, 2$. This gives

   $$z = 2cis90 = 2(0 + i1) = 2i,$$

   $$z = 2cis210 = 2(-\frac{\sqrt{3}}{2} - i\frac{1}{2}) = -\sqrt{3} - i,$$

   and

   $$z = 2cis330 = 2(\frac{\sqrt{3}}{2} - i\frac{1}{2}) = \sqrt{3} - i.$$

2. Course Notes, Problem 7.2.7. Take pains to make neat, clear diagrams.

   Solution.

   (a) Reflection. The line of reflection is the perpendicular bisector of segment $\overline{AB}$.

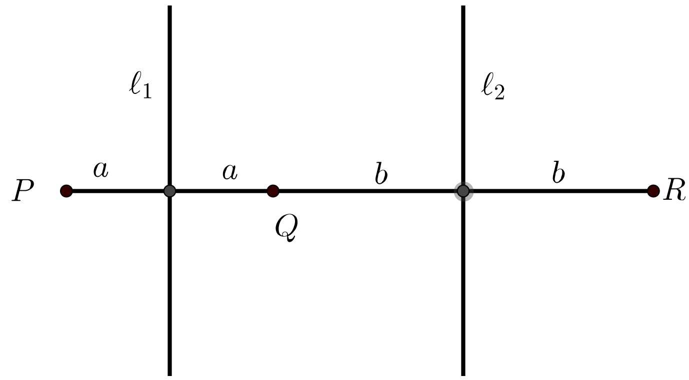
   (b) Rotation. Draw segments between two pairs of corresponding points, then construct the perpendicular bisectors to these segments. These bisectors will intersect at the center of rotation, C. Draw angle $\angle ACB$ to indicate the angle of rotation.

   (c) Glide Reflection. Draw segments between two pairs of corresponding points. Draw the line through the midpoints of these segments to get the line of reflection. Reflect the point A across this line to get the point A'. Draw vector $A'B$ to indicate the amount and direction of translation.

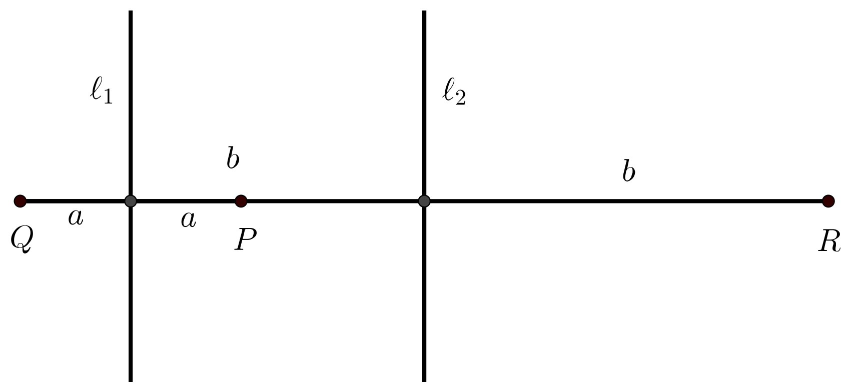
   (d) Translation. Draw vector AB to indicate the amount and direction of translation.

3. Prove that ℓ_1 and ℓ_2 are parallel lines, then the net effect of first reflecting across ℓ_1 and then reflecting across ℓ_2 is a translation in the direction perpendicular to the lines, directed from ℓ_1 towards ℓ_2, by an amount equal to twice the distance between the two lines.

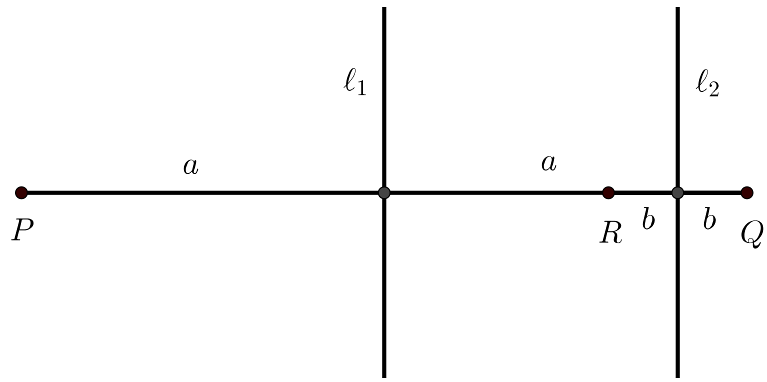
   Solution. Refer to the diagrams.



In the figure above a and b are both positive.



In the figure above a is negative and b is positive.



In the figure above a is positive and b is negative.

Let P be a point, Q be the reflection of P in ℓ_1, and R be the reflection of Q in ℓ_2. Let a be the directed distance from P to ℓ_1, where $a > 0$ if P is to the left of ℓ_1 and $a < 0$ if P is to the right of ℓ_1. Note that this directed distance is perpendicular to ℓ_1. Then the directed distance from ℓ_1 to Q is also a. Let b be the directed distance from Q to ℓ_2, where $b > 0$ if Q is to the left of ℓ_2 and $b < 0$ if Q is to the right of ℓ_2. Note that this directed distance is perpendicular to ℓ_2. Then the directed distance from ℓ_2 to R is also b. Thus the directed distance from P to R is $a + a + b + b = 2a + 2b$, and the directed distance from ℓ_1 to ℓ_2 is $a + b$.

4. (a) Consider the circles C_1 described by $(x - a_1)^2 + (y - b_1)^2 = c_1^2$ and C_2 described by $(x - a_2)^2 + (y - b_2)^2 = c_2^2$. Prove algebraically that C_1 and C_2 can share at most two points, and further, if they do share two different points P and Q, then the perpendicular bisector of the segment $\overline{PQ}$ is the line through the centers of the circles.

Solution. Expand the equations of the circles.

$$x^2 - 2a_1x + a_1^2 + y^2 - 2b_1y + b_1^2 = c_1^2,$$

$$x^2 - 2a_2x + a_2^2 + y^2 - 2b_2y + b_2^2 = c_2^2.$$

Subtract the second equation from the first.

$$(-2a_1 + 2a_2)x + (-2b_1 + 2b_2)y + a_1^2 - a_2^2 + b_1^2 - b_2^2 - c_1^2 + c_2^2 = 0.$$

This is a linear equation, and you can solve for one of the variables x, y (e.g., y) and substitute back into one of the original circle equations to get a quadratic equation in the other variable (e.g., x). Solving this equation by the quadratic formula yields at most two solutions for x, and each of these values for x gives one value for y from the linear equation.

If the two circles share two distinct points A, B, then $\overline{AB}$ is a chord of each circle, and its perpendicular bisector passes through the center of each circle.

(b) Let f be an isometry of the plane (not necessarily one of the four specific types we have been discussing). Let A, B, C be three noncollinear points. Show that if you know $f(A)$, $f(B)$, and $f(C)$, then you can determine $f(P)$ for any point. That is to say, f is uniquely determined by its action on any three particular noncollinear points.

Solution. First observe that since A, B, C are not collinear, and since isometries preserve distances between pairs of points, then $f(A), f(B), f(C)$ are also not collinear. Let $a = AP$, $b = BP$, and $c = CP$. Then we must have $a = f(A)f(P)$, $b = f(B)f(P)$, and $c = f(C)f(P)$. So the point $f(P)$ must lie on the intersection of three circles: C_1 centered at $f(A)$ with radius a, C_2 centered at $f(B)$ with radius b, and C_3 centered at $f(C)$ with radius c. We need to show that this common intersection cannot contain more than one point. But if it contained two points Q and R, then by the previous problem the centers of all three circles would lie on the perpendicular bisector of $\overline{QR}$ and thus be collinear, which would be a contradiction.

5. Consider the set of points S in the plane (a "strip") described by $S = \{(x, y) \in \mathbf{R}^2 : -1 \leq y \leq 1\}$. Carefully describe the set of all translations, rotations, reflections, and glide reflections that map every point in S back into S.

Solution.

(a) Translations by vectors $(p, 0)$ for any real number p. (This includes the identity map.)

(b) Rotations by 180 degrees about points $(p, 0)$ for any real number p.

(c) Reflections across vertical lines of the form $x = p$ for any real number p.

(d) Reflection across the horizontal line $y = 0$.

(e) Glide reflections across the horizontal line $y = 0$, with translation by $(p, 0)$ for any real number p.