MA 341 Class Notes

Contents

1 Wednesday, January 15

1. Passed out the syllabus and notes.
2. Worked on 1.1.1, the SETI puzzle.

2 Friday, January 17

Worked on 1.1.3–1.1.7.

3 Wednesday, January 22

1. Mentioned the programs GeoGebra, SketchUp, and POV-Ray.
2. Completed 1.1.3–1.1.7 and presented the beginning of Section 1.2.

4 Friday, January 24

Worked on problems similar to 1.2.2–1.2.5.

5 Monday, January 27

Went rather rapidly over some of the material in Sections 1.2–1.4, 1.6–1.10.

6 Wednesday, January 29

1. Collected Homework #1.

2. Worked on Section 2.2 of the course notes.

7 Friday, January 31

1. Assigned Homework #2.
2. Worked on 2.2.1–2.2.8 of course notes.
3. Demonstrated some visualizations of these results with the free programs SketchUp and Geogebra.

8 Monday, February 3

Class canceled due to bad weather.

9 Wednesday, February 5

1. Exam #1 will not be given on February 10 but will be moved to a later date.
2. Returned Homework #1 and offered some comments on solutions. In particular, for any problem for which you received a score below 4 points you may rewrite the problem and resubmit within a week to bring the score up to 4 points.
3. Mentioned that I posted a SketchUp file and two GeoGebra files.
4. Discussed some aspects of Homework #2, and decided to collect this on Friday rather than today.
5. Worked on Sections 2.2.9–2.2.12.

10 Friday, February 7

1. Collected Homework #2.

2. The mapping from the sphere to the plane described in problem #2 in the homework is called *stereographic projection*. It turns out that stereographic projection is angle preserving.

3. Discussed 2.2.1–2.2.12. In particular, with respect to POINTS, LINES, and the containment of POINTS in LINES:

   (a) Euclidean Geometry. Model 2.2.4 is isomorphic to model 2.2.1 via stereographic projection with respect to the point $(0, 0, 1)$ on the sphere to the plane $z = 0$. Model 2.2.10 is isomorphic to model 2.2.1 by intersecting the POINTS and LINES of 2.2.10 with the plane $z = 1$. Model 2.2.12 is isomorphic to model 2.2.4 by recognizing that equivalence classes of POINTS correspond to nonhorizontal lines through the origin (omitting the origin) and equivalence classes of LINES correspond to equations of nonhorizontal planes through the origin.

   (b) Spherical Geometry. Model 2.2.2 is not isomorphic to the other models.

   (c) Projective Geometry. Model 2.2.9 is isomorphic to model 2.2.3 by intersecting the POINTS and LINES of 2.2.9 with the unit sphere centered at the origin. Model 2.2.11 is isomorphic to model 2.2.9 by recognizing that equivalence classes of POINTS correspond to lines through the origin (omitting the origin) and equivalence classes of LINES correspond to equations of planes through the origin.

   (d) Hyperbolic Geometry. Model 2.2.5 is isomorphic to model 2.2.6 by vertical projection onto the plane $z = 0$. Model 2.2.5 is isomorphic to model 2.2.7 by stereographic projection with respect to the point $(0, 0, -1)$ and the plane $z = 0$. Model 2.2.5 is isomorphic to model 2.2.8 by stereographic projection with respect to the point $(1, 0, 0)$ and the plane $z = 1$.

   (e) Euclidean geometry can be thought of as embedded within projective geometry by placing Euclidean geometry in the plane $z = 1$. Then it makes sense to state that to go from Euclidean to projective geometry we are adding one point at infinity for each family of parallel lines, and one line at infinity containing all of these points at infinity.

11 Monday, February 10

Discussed Homework #2. Please resubmit in order to gain points.

12 Wednesday, February 12

1. Collected Homework #3. This is equivalent to a case of *Cramer's Rule*.
2. Discussed the definition and motivation for matrix multiplication.
3. Discussed Sections 2.4–2.6, and the comment on one way to remember matrix multiplication described after 2.7.2.
4. Exam #3 will cover material through today and will be on Wednesday, February 19.

13 Friday, February 14

1. Illustrated matrix multiplication, eigenvectors, and eigenvalues with GeoGebra. I will post the GeoGebra file to the course website.

2. Gave the formula for the inverse of a 2×2 matrix and illustrated how this can be used to solve two linear equations in two unknowns. Now that you know this, you can find a quicker solution to Homework #3.

3. Discussed the distance formula in the plane, and show how to derive it from the Pythagorean Theorem.

4. Started work on the problem of describing the set of points $P(x, y)$ such the distance from P to $A(4, 3)$ equals the distance from P to the line $y = 1$.

5. Homework solutions will be posted to the course website, and I am planning to send out a brief study guide for the exam on Wednesday.

14 Monday, February 17

1. Showed portions of videos and apps having to do with a sense of scale (posted to course website).

2. Finished the problem above (which is 3.3.27).

3. Worked on 3.3.23.

15 Wednesday, February 19

Exam #1.

16 Friday, February 21

1. Returned the exam and explained the grading curve.

2. Urged resubmissions for past homeworks by next Friday.

3. Recalled that we already discussed the distance formula and its derivation from the Pythagorean Theorem, as well as the solution to Problem 3.3.27.

4. Briefly discussed some problems:

   (a) 3.3.1. Most likely straight line distance makes sense (unless there is an animal in there with you?).

   (b) 3.3.2 and 3.3.11. Now you probably must use roads if you are driving, and distances (or times) might change if there are some one way roads. There is an area in graph theory that considers the problem of algorithms for shortest distances in networks, and undoubtedly the GPS takes advantage of these algorithms.

   (c) 3.3.3. Again, probably road distance is more important than straight line distance.

   (d) 3.3.4. If you are flying, great circle distance now plays a role.

   (e) 3.3.6. Derived the formulas $\int_x \sqrt{1+(\frac{dy}{dx})^2}dx$ and $\int_t \sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}dt$ from the distance formula. This can be used to calculate the length of a catenary, the shape of the St. Louis arch.

   (f) 3.3.7. There are challenges with determining "inaccessible" distances. For example, distances to near stars can be computed via measuring the angles to these stars from the earth at two opposite times of the year, and using triangle trig formulas. (Look up "parallax".)

   (g) 3.3.9. Distance may depend more on how many intermediate nodes there are between two computers on the internet.

   (h) 3.3.10. When transmitting a word with 0-1 bits, some bits may be flipped, and the distance between the correct and the incorrect word might be the number of such flipped bits. One can append additional "check bits" to help determine how many bits have been flipped, and to correct a certain number of them. (Look up "Hamming code.) This kind of problem also relates to the question of how far apart two genetic codes are.

   (i) 3.3.15. This problem generates Voronoi regions. Find a functioning applet to demonstrate! The program Wingeom has a Voronoi window, `http://math.exeter.edu/rparris/wingeom.html`.

(j) 3.3.22. This is the problem of finding “Steiner points”. Look this up for some examples. This problem can be solved with two sheets of plexiglass separated by four short rods and immersed in soapy water. When you lift it out, the soap bubbles may form the correct shape.

5. Solved Problem 3.3.23 in general, once with a geometric solution, and once with a calculus solution. In both cases we saw that the critical point on the river needed to be chosen so that two particular angles were congruent. This and problem 3.3.24 are interesting in the context of the principle in physics that light follows paths of least time.

6. I asked that everyone solve Problem 3.3.24 by Monday.

17 Monday, February 24

1. Discussed Homework #4 but postponed collection until Wednesday (and I will add another problem). In this problem Snell's Law is derived, which relates to the refraction of light as it passes from one medium into another.

2. Started working on Problem 3.3.25.

18 Wednesday, February 26

1. Discussed some of the second problem of Homework #4, and illustrated a construction with GeoGebra.

2. Discussed Section 3.8 on the Triangle Inequality.

3. Mentioned the notion of *betweenness*: If A, B, and C are three distinct points on a line, then B is between A and C if $AB + BC = AC$.

19 Friday, February 28

1. Discussed the distance formula in $\mathbf{R}^3$ and how to derive it from the distance formula in $\mathbf{R}^2$ and the Pythagorean Theorem.
2. Mentioned Theorem 4.2.4, but did not prove this.
3. Discussed Section 6.2 and the beginning of Section 6.3.

20 Monday, March 3

UK classes canceled due to inclement weather.

21 Wednesday, March 5

1. Went over and collected homework. In particular, we derived the formula for the reflection of a point in a line with equation $ax + by = c$. This formula looks nicer if we first divide through the equation of the line by $\sqrt{a^2 + b^2}$, so that we can assume that $a^2 + b^2 = 1$.

2. Modeled the rotation and revolution of a planet about a sun—see the posted GeoGebra file. In particular, if a planet rotates counterclockwise around its axis d times while revolving once counterclockwise around the sun, then the inhabitants will experience only $d - 1$ days.

3. Mentioned that you can calculate $\sin(x)$, $\cos(x)$, and e^x using Taylor series. More on this later.

22 Friday, March 7

1. Reminder about the take-home exam.

2. Midterm grades will be posted next week—please submit any missing homework ASAP.

3. Played with Scott Kim's ambigram puzzles — there will be a link on the course website.

4. Showed some of the other links from Scott Kim's talk on symmetry — see the link on the course website.

5. Briefly introduced the notions of translation, rotation, and reflection. Sometimes reflection in the plane is carried out by "flipping" or rotating through three-dimensional space. It is possible to make sense of the notion of reflecting across a plane in three-dimensional space by rotating through four-dimensional space.

6. Experimented with the result of performing a reflection across a line, followed by a reflection across a parallel line. This appeared to be equivalent to a translation.

23 Monday, March 10

1. Discussed how to do problems like 6.5.2–6.5.4.

2. Illustrated with GeoGebra that multiplication by i is equivalent to a 90 degree rotation counterclockwise, and hence that $i^2 = -1$ makes sense geometrically.

3. Discussed the geometric definitions of complex number addition and multiplication (see Problem 6.6.3), but so far without justification. Explained why restriction of this definition to the real number axis corresponds to multiplication in the real case.

4. Solved $z^2 = 1$ (two solutions) and $z^3 = 1$ (three solutions), and sketched how to solve $z^3 = i$.

24 Wednesday, March 12

1. Discussed 6.6.2 (but did not discussion division, though this is not hard now), 6.6.3, 6.6.5, 6.6.9–6.6.10.

2. Used complex numbers to derive a formula, expressed in terms of complex numbers, to rotate a point z about another point w by a counterclockwise angle of θ:

$$f(z) = (\mathrm{cis}\theta)(z - w) + w.$$

25 Friday, March 14

1. Collected Exam #2.

2. Discussed 7.1.1–7.1.12 and 7.2.1–7.2.3. We did not completely finish 7.2.2 and 7.2.3, and I encouraged everyone to play with these with GeoGebra over the break.

26 Monday, March 24

1. Discussed 7.2.3 (and you should now be able to do 7.2.7), 7.2.8, mentioned 7.2.9, 7.2.10, and 7.2.12. Gave an example to show that the isometry might not be determined by the action on three collinear points.

27 Wednesday, March 26

1. Finished up the matrix formulas for translations, rotations, reflections, and glide reflections.

28 Friday, March 28

1. Proved that a reflection (as described by its matrix) is indeed an isometry.
2. Proved that every isometry is a composition of at most three reflections, using the result that an isometry is determined by its action on three noncollinear points.

29 Monday, March 31

1. Proved that the composition of two reflections across parallel lines ℓ_1, ℓ_2 is a translation by an amount equal to twice the distance between these two lines in the direction perpendicular to these lines directed from ℓ_1 towards ℓ_2. Special case: $\ell_1 = \ell_2$, in which case the composition is the identity map.

2. Proved that the composition of two reflections across nonparallel lines ℓ_1, ℓ_2 crossing at point P is a rotation about P by an angle equal to twice the angle between these two lines directed from ℓ_1 towards ℓ_2. Special case: $\ell_1 \perp \ell_2$, in which case the composition is a rotation by 180 degrees around the point P of intersection; equivalently reflection through the point P. In this case the order in which the reflections occurs does not matter.

3. Proved that every glide reflection is the composition of three reflections.

4. Proved that the composition of three reflections in parallel lines is again a reflection, and showed how to find the net line of reflection.

5. Began to analyze the composition of three reflections in the case that the three lines are not mutually parallel.

30 Wednesday, April 2

1. Discussed some of the homework problems—I will accept homework in my mailbox before Friday.
2. Worked a bit more on the composition of three reflections.
3. There will be no class on Friday due to the undergraduate research conference at UK.

31 Friday, April 4

No class today due to the undergraduate research conference at UK.

32 Monday, April 7

1. Finished the proof that three reflections is either a single reflection or a glide reflection.
2. Use the "algebra of reflection lines" to compose two rotations.
3. We will shortly turn to some geometry software, with the intent of some particular assignments that will count toward Exam #3 and the Final Exam.

33 Wednesday, April 9

1. Discussed homework.

2. Defined *border patterns* and proved that there are seven different symmetry types (see Problem 7.7.6).

34 Friday, April 11

Introduction to POV-Ray.

35 Monday, April 14

Some more features of POV-Ray.

36 Wednesday, April 16

1. Worked on getting the coordinates of an icosahedron, following the guidelines in "Coordinates for the Platonic Solids" posted on the course website. I also used Wingeom (which does not run in the Mac OS) to illustrate the construction—see the files on the website.

2. Worked on using these coordinates and the set of triangles to prepare input for POV-Ray to display the icosahedron. I will write up a specific assignment based on this to be counted toward Exam #3.

37 Friday, April 18

Gave an introduction to OpenSCAD and SketchUp, both of which can be used to prepare models for 3D printing. I will write up a specific assignment based on this to be counted toward the Final Exam.

38 Monday, April 21

Worked on the sequence of creating a shape in OpenSCAD, exporting it to MakerWare, and then creating the file for the 3D printer.

39 Wednesday, April 23

Introduction to the six four-dimensional regular solids. Today we examined the 5-cell, 8-cell, and (briefly) the 16-cell. See the extra course notes on Visualizing Math posted to the course website.

40 Friday, April 25

Discussed the 16-cell and the 24-cell, and briefly introduced the 120-cell and the 600-cell. In dimensions 5 and higher, though, there are only three regular polyhedra—the analogs of the tetrahedron, cube, and octahedron.