Geometry Notes

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Think Deeply of Simple Things
Motto of the Ross Program at the Ohio State University

Contents

1 Axiomatic Systems

1.1 Initial Explorations

We begin with a puzzle based on a creation of Ivan Bell. Suggestion: First watch the trailer for the movie *Contact*: `http://www.youtube.com/watch?v=SRoj3jK37Vc`.

Problem 1.1.1 (SETI Puzzle) The following message has been received from outer space. You believe it is from an alien intelligence in our solar system with a sincere desire to communicate. What do you make of it? The message contains mixtures of 24 different symbols, which we will represent here, for convenience, by the letters from A through Z (omitting O and X). "(Each symbol is presumably radioed by a combination of beeps, but we need not be concerned with those details.) The punctuation marks are not part of the message but indications of time lapses. Adjacent letters are sent with short pauses between them. A space between letters means a longer pause. Commas, semicolons, and periods represent progressively longer pauses. The longest time lapses come between paragraphs, which are numbered for the reader's convenience; the numbers are not part of the message." To get you started, the first paragraph is merely a transmission of the 24 symbols to be used in the rest of the message.

1. A. B. C. D. E. F. G. H. I. J. K. L. M. N. P. Q. R. S. T. U. V. W. Y. Z.
2. A A, B; A A A, C; A A A A, D; A A A A A, E; A A A A A A, F; A A A A A A A, G; A A A A A A A A, H; A A A A A A A A A, I; A A A A A A A A A A, J.
3. A K A L B; A K A K A L C; A K A K A K A L D. A K A L B; B K A L C; C K A L D; D K A L E. B K E L G; G L E K B. F K D L J; J L F K D.
4. C M A L B; D M A L C; I M G L B.
5. C K N L C; H K N L H. D M D L N; E M E L N.
6. J L AN; J K A L AA; J K B L AB; AA K A L AB. J K J L BN; J K J K J L CN. FN K G L FG.
7. B P C L F; E P B L J; F P J L FN.

8. F Q B L C; J Q B L E; FN Q F L J.
9. C R B L I; B R E L CB.
10. J P J L J R B L S L ANN; J P J P J L J R C L T L ANNN. J P S L T; J P T L J R D.
11. A Q J L U; U Q J L A Q S L V.
12. U L WA; U P B L WB; AWD M A L WD L D P U. V L WNA; V P C L WNC. V Q J L WNNA; V Q S L WNNNA. J P EWFGH L EFWGH; S P EWFGH L EFGWH.
13. GIWIH Y HN; T K C Y T. Z Y CWADAF.
14. D P Z P WNNIB R C Q C.

Problem 1.1.2 (SETI Puzzle Follow Up) What do you think of the statement that "Mathematics is the only truly universal language"?

Problem 1.1.3 (Carrollian System I) Contact has been established with an alien race (the Carrollians) and they convey the following information to you about a mathematical structure of interest to them.

A. There is a finite number of toves.

B. There is a finite number of borogoves.

C. Given any borogove there are exactly two different toves that gimble with it.

Prove that the number of toves that gimble with an odd number of borogoves is even.

Problem 1.1.4 (Carrollian System I Follow Up) What is the meaning of toves and borogoves? What does gimble mean? What representations, if any, did you create while working on this problem?

Problem 1.1.5 (Handshaking) At a recent conference, various pairs of people shook hands. Prove that the number of people who shook hands an odd number of times is even.

Problem 1.1.6 (Graphs) A *graph* $G = (V, E)$ consists of a finite set V of *vertices* and a finite set E of *edges*. Assume there are no loops, so each edge joins two distinct vertices. The *degree* of a vertex is the number of edges joined to it. Prove that the number of vertices of odd degree is even.

Problem 1.1.7 (Lines) Consider a finite collection $\mathcal{L}$ of lines in the plane with the property that no three pass through a common intersection point. Let $\mathcal{P}$ be any subset of the collection of all the various intersection points of these lines. Prove that the number of lines of $\mathcal{L}$ containing an odd number of points in $\mathcal{P}$ is even.

Problem 1.1.8 (Polyhedra) Join together a finite collection of convex polygons edge to edge to enclose a region of space, with two polygons meeting at each edge. Prove that the number of polygons with an odd number of sides is even.

Problem 1.1.9 (Labeled Triangles) Draw a triangle T and label its vertices 1, 2, and 3. Subdivide the triangle into smaller triangles, introducing new vertices if you wish, which can also be along the edges of T. Label each new vertex 1, 2, or 3, any way you want, with the following restriction: You can only use labels 1 and 2 for the new vertices along the original "12 edge" of T, and similarly for the other edges of T. Prove that there must be a "123 triangle."

1.2 Features of Axiomatic Systems

One motivation for developing axiomatic systems is to determine precisely which properties of certain objects can be deduced from which other properties. The goal is to choose a certain fundamental set of properties (the *axioms*) from which the other properties of the objects can be deduced (e.g., as *theorems*). Apart from the properties given in the axioms, the objects (nouns) and relations (verbs) are regarded as *undefined*.

As a powerful consequence, once you have shown that any particular collection of objects satisfies the axioms *however unintuitive or at variance with your preconceived notions these objects may be*, without any additional effort you may immediately conclude that all the theorems must also be true for these objects.

We want to choose our axioms wisely. We do not want them to lead to contradictions; i.e., we want the axioms to be *consistent*. We also strive for economy and want to avoid redundancy—not assuming any axiom that can be proved from the others; i.e., we want each axiom to be *independent* of the others so that the axiomatic system as a whole is *independent*. Finally, we may wish to insist that we be able to prove or disprove any statement about our objects from the axioms alone. If this is the case, we say that the axiomatic system is *complete*.

We can verify that an axiomatic system is consistent by finding a *model* for the axioms—a choice of objects and relations that satisfy the axioms.

We can verify that a specified axiom is independent of the others by finding two models—one for which all of the axioms hold, and another for which the specified axiom is false but the other axioms are true.

We can verify that an axiomatic system is *complete* by showing that there is essentially only one model for it (all models are *isomorphic*); i.e., that the system is *categorical*.

Problem 1.2.1 Consider the system in Problem 1.1.3.

1. Explain why you know that the system is consistent.

2. Determine whether or not each axiom of the system is independent.

3. Determine whether or not the system is categorical.

Problem 1.2.2 Consider the following axioms for a certain committee structure (though we could just as well use “toves” for people, “borogoves” for committees, and “gimble” for membership):

A. There are exactly four people.

B. There are exactly seven committees.

C. Each committee consists of exactly two people.

D. No two committees have the same set of people as members.

1. Is this system consistent?
2. If so, which axioms are independent of the others?
3. Is this system categorical?

Problem 1.2.3 Consider the following axioms for a certain committee structure:

A. There are exactly four people.

B. There are exactly six committees.

C. Each committee consists of exactly two people.

D. No two committees have the same set of people as members.

E. Each person serves on exactly three committees.

1. Is this system consistent?
2. If so, which axioms are independent of the others?
3. Is this system categorical?

Problem 1.2.4 Consider the following axioms for a certain committee structure:

A. There are exactly four people.

B. There are exactly five committees.

C. Each committee consists of exactly two people.

D. No two committees have the same set of people as members.

1. Is this system consistent?
2. If so, which axioms are independent of the others?
3. Is this system categorical?

Problem 1.2.5 Consider the following axioms for a certain committee structure:

A. There are exactly four people.

B. There are exactly four committees.

C. Each committee consists of exactly two people.

D. No two committees have the same set of people as members.

1. Is this system consistent?
2. If so, which axioms are independent of the others?
3. Is this system categorical?

Problem 1.2.6 (Carrollian System II) Consider the following system:

A. There is a finite number of toves.

B. There is a finite number of borogoves.

C. Given any borogove there are exactly two different toves that gimble with it, and further this is the only borogove that these two toves gimble with.

D. Every tove gimbles with a different number of borogoves.

1. Is this system consistent?

2. What are the implications for the settings of Problems 1.1.5, 1.1.6, 1.1.7, and 1.1.8?

Problem 1.2.7 (Carrollian System III) Consider the following system:

A. Given any two different toves, there is exactly one borogove that gimbles with both of them.

B. Given any two different borogoves, there is exactly one tove that gimbles with both of them.

C. There exist four toves, no three of which gimble with a common borogove.

D. There exists a borogove that gimbles with exactly three toves.

1. Is this system consistent?

2. Determine whether or not each axiom of the system is independent.

3. Determine whether or not the system is categorical.

Problem 1.2.8 Show that the following interpretation is a valid model for the system in Problem 1.2.7. (1) Toves are triples of the form (x, y, z) where each of x, y, and z are 0 or 1, and not all of them are zero. (2) Borogoves are triples of the form (x, y, z) where each of x, y, and z are 0 or 1, and not all of them are zero. (3) A tove (x_1, y_1, z_1) gimbles with a borogove (x_2, y_2, z_2) if $x_1x_2 + y_1y_2 + z_1z_2$ is even.

Problem 1.2.9 Drop axiom (D.) from the system in Problem 1.2.7 and assume instead that that there exists a borogove that gimbles with exactly $q+1$ toves, $q \geq 2$. Prove that every borogove gimbles with exactly $q+1$ toves, every tove gimbles with exactly $q+1$ borogoves, there is a total of q^2+q+1 toves, and there is a total of q^2+q+1 borogoves.

Problem 1.2.10 Drop axiom (D.) from the system in Problem 1.2.7 and assume instead that that there exists a borogove that gimbles with exactly $q+1$ toves, $q \geq 2$.

1. Prove that there exist four borogoves, no three of which gimble with a common tove.
2. Prove that there exists a tove that gimbles with exactly $q+1$ borogoves.

(Note that this means that the terms "tove" and "borogove" are interchangeable in every theorem of this system.)

Problem 1.2.11 Drop axiom (D.) from the system of Problem 1.2.7 and find a model containing exactly 13 toves.

1.3 Finite Projective Planes

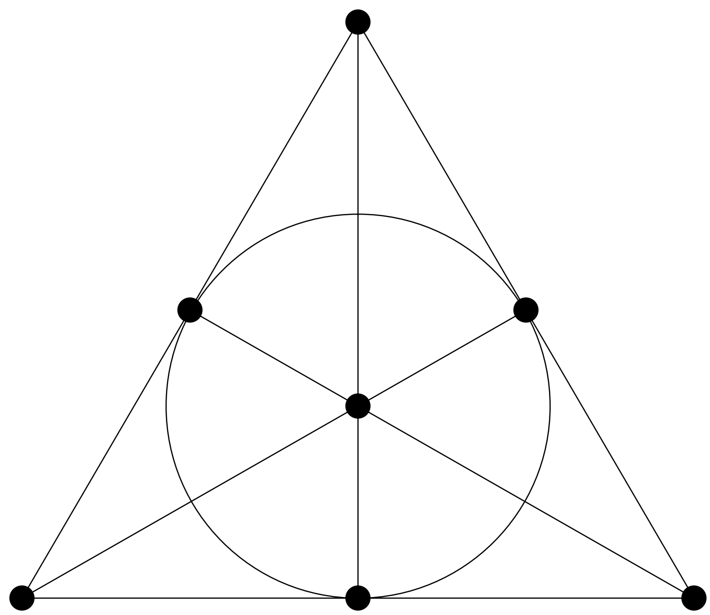
Dropping axiom (D.) from the system in Problem 1.2.7 (and, if you wish, replacing the words "toves" with "points", "borogoves" with "lines," and "gimbles with" with "is incident to"), we define structures called *finite projective planes*. I have a game called *Configurations* that is designed to introduce the players to the existence, construction, and properties of finite projective planes. When I checked in January 2014 the game was available from WFF 'N PROOF Games for Thinkers, `http://wffnproof.com/home`, for a cost of $25.00.

Here are examples of some problems from this game:

Problem 1.3.1 In each box below write a number from 1 to 7, subject to the two rules: (1) The three numbers in each column must be different; (2) the same pair of numbers must not occur in two different columns.

	Col 1	Col 2	Col 3	Col 4	Col 5	Col 6	Col 7
Row 1							
Row 2							
Row 3							

Problem 1.3.2 Use the solution to the above problem to label the seven points of the following diagram with the numbers 1 through 7 so that the columns of the above problem correspond to the triples of points in the diagram below that lie on a common line or circle.



Problem 1.3.3 Play the game of *SET*. (This game can be found, for example, on `http://www.amazon.com` under the name "SET Game" by SET Enterprises, Inc., and there is also a SET app for the iPad.) An online daily SET puzzle of the day can be found here: `http://www.setgame.com/set/puzzle_frame.htm`. Find some ways to think of this game as a model for a set of axioms for points and lines (and planes?).

1.4 Kirkman's Schoolgirl Problem

Problem 1.4.1 Solve the following famous puzzle proposed by T. P. Kirkman in 1847:

> *A school-mistress is in the habit of taking her girls for a daily walk. The girls are fifteen in number, and are arranged in five rows of three each, so that each girl might have two companions. The problem is to dispose them so that for seven consecutive days no girl will walk with any of her school-fellows in any triplet more than once.* (Ball and Coxeter, *Mathematical Recreations and Essays*, University of Toronto Press, 1974, Chapter X.)

1.5 Categoric and Complete

In the book *Geometry: An Introduction* by Günter Ewald, a different definition of "complete" is given. An axiomatic system is called categorical if all models for it are isomorphic. An axiomatic system is called *complete* if no model for it can be extended by adding "new objects in such a way that all previous relations are carried over and such that all previous axioms remain true in the enlarged system." Here are two exercises from that book:

Problem 1.5.1 Show that the axioms of a group together with the following axiom are complete but not categorical: "The group contains precisely four elements."

Problem 1.5.2 Show that the axioms of a group together with the following axiom are categorical but not complete: "The group has infinitely many elements and consists of all powers of a single group element."

1.6 Other Axiomatic Systems

For perhaps understandable reasons most non-mathematics majors associate axiomatic systems exclusively with the realm of geometry, not realizing its all-pervading presence in mathematics.

Problem 1.6.1 Look up examples of other axiomatic systems. Here are some examples:

1. Equivalence Relations
2. Sets
3. Integers
4. Real numbers
5. Groups
6. Rings
7. Fields
8. Vector spaces
9. Metric spaces
10. Topological spaces
11. Probability spaces
12. Graphs
13. Partially ordered sets
14. Matroids

1.7 Some Milestones in Geometry (Very Incomplete)

1. Ancient. Examples:
   (a) Egyptian, c. 2000 BC. Some Pythagorean triples. Estimates for area of circle. Volumes of truncated pyramids.
   (b) Babylonian, c. 1900–1600 BC. Pythagorean triples. Estimates for area of circle. Base 60 system leading to our use of 360 degrees in a circle.
2. Greek
   (a) Thales, 635–543 BC. Demonstrative mathematics.
   (b) Pythagoras, 582–496 BC. Pythagorean Theorem, irrational quantities.
   (c) Plato, 427–347 BC. School—"Let none ignorant of geometry enter here." Straightedge and compass constructions.
   (d) Euclid, c. 325–265 BC. *The Elements of Geometry.* Perhaps the second most widely published book in human history. Logical organization via an axiomatic system.
   (e) Archimedes, 287–212 BC. One of the greatest mathematicians in human history. Area of circle, volume and surface area of sphere. Developer of *The Method.*
3. India. Numerous contributions.
4. China. Numerous contributions.
5. Islam, 640 AD onward. In addition to developing new mathematics, Islamic centers of learning preserved Greek mathematics, which declined in Europe.
6. Filippo Brunelleschi (1404–1472), Johannes Kepler (1571–1630), Gérard Desargues (1591–1661). Projective geometry and perspective drawings.
7. Cartesian coordinates and analytic geometry. René Descartes, 1596–1650, and Pierre de Fermat, 1601–1665.
8. Calculus. Isaac Newton, 1642–1727, and Gottfried Wilhelm von Leibniz, 1646–1716. Areas under curves.
9. Non-Euclidean geometry. Carl Friedrich Gauss (1777–1855), János Bolyai (1802–1860), Nikolai Lobachevsky (1792–1856), Bernhard Riemann (1826–1866). Proved the independence of Euclid's parallel postulate by finding models of non-Euclidean geometry. Riemann's geometry provided the basis for the geometry of Einstein's general relativity.

10. Klein, 1849–1925. Non-Euclidean geometry. Study of geometries in the context of transformations.

11. David Hilbert, 1862–1943. Axiomatic system for Euclidean Geometry presented in *Foundations of Geometry*.

12. Jakob Steiner (1796–1863), Thomas Kirkman (1806–1895), Gino Fano (1871–1952). Finite geometries.

13. George Birkhoff, 1884–1944. Axiomatic system for Euclidean geometry with ruler and angle measurement axioms.

14. Committee of Ten, 1892. Made recommendations regarding the high school curriculum.

15. School Math Study Group, 1958–1977. Created in the wake of Sputnik, resulted in "New Math" movement.

16. National Council of Teachers of Mathematics *Principles and Standards for School Mathematics*, 2000.

17. *Common Core State Standards for Mathematics*, 2010.

1.8 Euclidean Geometry

Problem 1.8.1 Here is a website for Euclid's *Elements:* `http://aleph0.clarku.edu/~djoyce/java/elements/elements.html`. Look through this and do some research on the *Elements.*

1. What is its history and significance?
2. Summarize the content of each of the thirteen books. Look for your favorite theorems! What geometrical results appear to be missing?
3. Use GeoGebra to make some of the constructions from Book I.
4. What is the parallel postulate?
5. What is the first proposition in Book I that relies upon the parallel postulate?
6. Describe the historical development ultimately leading to the proof of the independence of the parallel postulate.
7. What are some implicit assumptions made by Euclid that are not explicitly spelled out?

Problem 1.8.2 You can find and download the book *Foundations of Geometry* by David Hilbert from `http://books.google.com`.

1. What is the significance of this book?
2. Compare the axioms in this book to the postulates of Euclid.

Problem 1.8.3 Here is the website for the School Math Study Group (SMSG) high school texts: `http://onlinebooks.library.upenn.edu/webbin/book/lookupname?key=School%20Mathematics%20Study%20Group`. Study Units 13 and 14.

1. Compare the axioms in this book to those of Hilbert.

2. Which results lead to the definition of coordinates for points in the plane?
3. Which results lead to the ability to define trigonometric ratios?

Problem 1.8.4 Here is the website for the Common Core State Standards for Mathematics: `http://www.corestandards.org/the-standards`. Read the section on geometry in high school.

1. How does this compare to the SMSG material?
2. What topics do you feel most comfortable with? Least comfortable with?

1.9 Personal Musings

I believe that there are several different viewpoints from which people (and mathematicians) may think about axiomatic systems. Let me elaborate a bit with respect to views of geometry.

Viewpoints:

1. We have an image in our minds of geometrical objects, and we regard geometry as a (large) collection of facts and properties, not necessarily organized in any particular way.
2. We have an image in our minds of geometrical objects, and we organize the facts from simplest to more complicated, with later facts provable from earlier facts. The simplest facts are regarded as "self-evident" and therefore exempt from proof.
3. We have an image in our minds of geometrical objects, and we organize facts as in (2), referring to the simplest, unproven facts, as the axioms. We recognize that despite our mental image, we cannot use any properties in our proofs that are not derivable from the axioms.
4. We have an image in our minds of geometrical objects, and we organize facts as in (3). We further recognize that despite our mental image, objects and relations specified in the axioms (such as "point", "line", "incidence", "between") are truly undefined, and that therefore in any other model in which we attach an interpretation to the undefined objects and relations for which the axioms hold, all subsequent theorems will hold also.
5. We have an image in our minds of geometrical objects, and we organize facts as in (4). But we further become familiar with and work with alternative models, and models of alternative axiom systems.
6. We have an image in our minds of geometrical objects, and we organize facts as in (5). But we fully recognize that all proofs in an axiom system are completely independent of any image in anyone's mind. (If we receive a set of axioms from an alien race about its version of geometry, we realize that we can prove the theorems without knowing what is in the minds of the aliens.)
7. We regard the formal system of axioms and theorems as all that there is—there is "nothing more out there" in terms of mathematical reality. (The aliens may in fact

have nothing in their heads but operate formally with the symbols and procedures of formal logic.)

I distinctly remember the struggle I had in high school of trying to understand the teacher's explanation of viewpoints (3) and (4), but I don't believe I really understood viewpoints (4) and (5) until college. I believe that I presently operate in practice from viewpoints (5) and (6). Computer automated proof systems (but not necessarily those who use them) operate from viewpoint (7).

1.10 Gödel's Theorems

Gödel's work had profound implications of what could or could not be proved in mathematics. Listen to the podcast from the BBC series In Our Time, `http://www.bbc.co.uk/programmes/b00dshx3`. You can also find suggestions for further reading at that website, including the Pulitzer prize winning book *Gödel, Escher, Bach: An Eternal Golden Braid* by Douglas Hofstadter. Here are some questions relating to the podcast:

1. What are axioms?
2. What are theorems?
3. Where does Euclid prove that there are infinitely primes?
4. What is the historical origin of these formal systems?
5. What is Euclid's Elements?
6. In what ways is mathematics different from other disciplines?
7. What are non-Euclidean geometries?
8. Why were people worried about them?
9. What did Cantor prove about infinities?
10. What did David Hilbert do in 1900?
11. What is the Hilbert program?
12. How did the view of "geometry" change?
13. What is a formalist?
14. What was Hilbert's problem about the theory of numbers (arithmetic)?
15. What did Hilbert do in the realm of Euclidean geometry?
16. What is a complete and decidable system?
17. What role did the study of set theory play?

18. What is a set?
19. What was Cantor trying to do with respect to sets?
20. Who laid down a formal system of axioms for sets?
21. What is Russell's paradox?
22. What was Frege working on?
23. What is the barber paradox?
24. What did Gödel in 1931?
25. How did this destroy Hilbert's vision?
26. What are Gödel's two theorems (consistency, incompleteness)?
27. Why were these results disturbing?
28. Who immediately understood the significance of Gödel's lecture?
29. What is the analogy of the game board?
30. What were Hilbert's reactions?
31. What was Russell's reaction?
32. What was Zermelo's reaction?
33. What is the Bourbaki group? What was their reaction?
34. What are some of the differences between Hilbert and Gödel?
35. What did Gödel prove in general relativity theory?
36. How did Gödel prove his incompleteness theorem?
37. How much impact did this have on working mathematicians?
38. What was Hilbert's first question?
39. What did Cantor ask?
40. What did Cohen prove?

41. What is the Continuum Hypothesis?

42. How did Hermann Weyl feel?

43. Were doubts raised about the consistency of Zermelo-Frankl set theory, Peano arithmetic?

44. What is the Goldbach conjecture? (Every even number the sum of two primes.)

45. What is the difference between proof and truth?

46. What is special about a fifth order polynomial equation?

47. What is the level of critical complexity?

48. Is there a Gödel theorem for Euclidean geometry?

49. What is the difference between Hilbert's and Gödel's view of mathematics?

50. What is the relevance of Gödel's Theorems to computers and computer proof?

51. What is Turing's halting problem?

52. What were the impacts on other disciplines?

53. What was the reaction of Freeman Dyson?

54. How do you feel about mathematics after hearing about what Gödel did?

2 Points, Lines, and Incidence

2.1 Incidence Axioms

Axioms for points and lines often use the relation *incidence,* as in "point A is incident to line L." But the axioms soon imply that we can regard lines as certain sets of points, so that is what we will do for now. Here is an axiom for points and lines.

Axiom 2.1.1 (SMSG Postulate 1) *Given any two different points, there is exactly one line which contains both of them.*

Notation: The line containing the points P and Q is denoted $\overleftrightarrow{PQ}$.

We already have a simple theorem.

Theorem 2.1.2 (SMSG Theorem 3-1) *Two different lines intersect in at most one point.*

Note: To say that the lines *intersect* is really to say they have a *nonempty intersection.*

Problem 2.1.3 Prove this theorem.

2.2 Geometrical Worlds

Problem 2.2.1 Here are some geometrical "worlds." In each case we make certain choices on what we will call POINTS and LINES. (I capitalize these words as a reminder these may not appear to be our "familiar" points, lines and planes.) In each case you should begin thinking about what properties hold for our choice of POINTS and LINES. In particular,

1. Is it true or false that given any two different POINTS, there is exactly one LINE that contains both of them?
2. Is it true or false that for any given LINE and any given POINT not on that LINE, there is a unique LINE containing the given POINT that does not intersect the given LINE?

It would be helpful for experimentation to have some spherical surfaces to draw on, such as (very smooth) tennis balls, ping-pong balls, oranges or Lénárt spheres.

2.2.1 The Analytical Euclidean Plane: $\mathbf{E}^2$

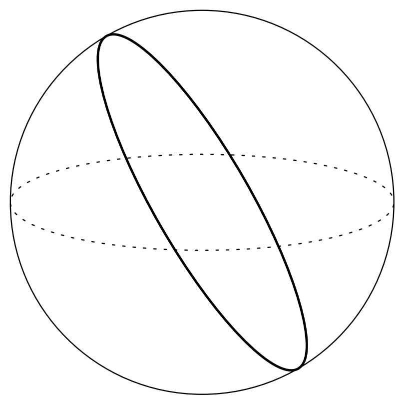
POINTS: Ordered pairs (x, y) of real numbers; i.e., elements of $\mathbf{R}^2$.

LINES: Sets of points that satisfy an equation of the form $ax + by + c = 0$, where a, b and c are real numbers; and further a and b are not both zero.

2.2.2 The Sphere: $\mathbf{S}^2$

POINTS: All points in $\mathbf{R}^3$ that lie on a sphere of radius 1 centered at the origin.

LINES: Great circles on the sphere (circles that divide the sphere into two equal hemispheres).



2.2.3 The Paired Sphere

(Note: This is not a standard name.)

POINTS: All pairs of points in $\mathbf{R}^3$ that lie on a sphere of radius 1 centered at the origin and are opposite each other.

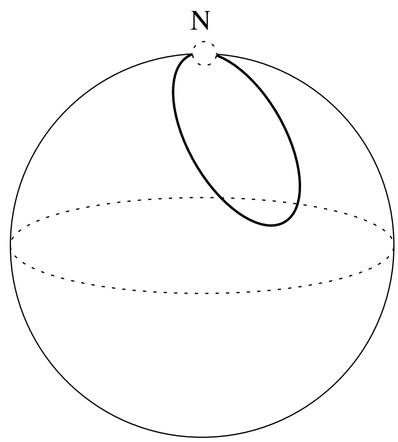
LINES: Great circles on the sphere.

2.2.4 The Punctured Sphere

(Note: This is not a standard name.)

POINTS: All points in $\mathbf{R}^3$ that lie on a sphere of radius 1 centered at the origin, with the exception of the point $N = (0, 0, 1)$ (the "North Pole"), which is excluded.

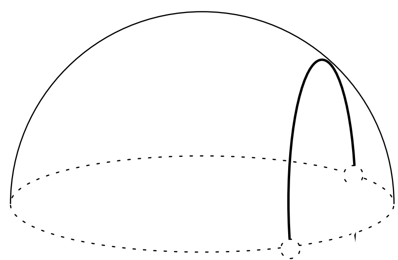
LINES: Circles on the sphere that pass through N, excluding the point N itself.



2.2.5 The Open Hemisphere

POINTS: All points in $\mathbf{R}^3$ that lie on the upper hemisphere of radius 1 centered at the origin and have strictly positive z-coordinate. (So the "equator" of points with z-coordinate equaling 0 is excluded.)

LINES: Semicircles (not including endpoints) on this hemisphere that are perpendicular to the "equator".



2.2.6 The Klein Disk

POINTS: All points in $\mathbf{R}^2$ that lie strictly in the interior of the circle of radius 1 centered at the origin.

LINES: Chords of the circle, excluding endpoints.

2.2.7 The Poincaré Disk: $\mathbf{H}^2$

POINTS: All points in $\mathbf{R}^2$ that lie strictly in the interior of the circle C of radius 1 centered at the origin.

LINES: Points of $\mathbf{H}^2$ that lie on circles intersecting C in right angles, as well as diameters (excluding endpoints) of C.

2.2.8 The Upper Half Plane

POINTS: All points (x, y) in $\mathbf{R}^2$ for which $y > 0$.

LINES: Points of the upper half plane that lie on circles intersecting the x-axis in right angles, or that lie on vertical lines.

2.2.9 The Projective Plane: $\mathbf{P}^2$

POINTS: All ordinary lines in $\mathbf{R}^3$ that pass through the origin.

LINES: All ordinary planes in $\mathbf{R}^3$ that pass through the origin.

2.2.10 The Affine Plane: $\mathbf{A}^2$

POINTS: All ordinary nonhorizontal lines in $\mathbf{R}^3$ that pass through the origin.

LINES: All ordinary nonhorizontal planes in $\mathbf{R}^3$ that pass through the origin.

2.2.11 The First Vector Plane

(Note: This is not a standard name.)

POINTS: Ordered triples (x, y, z) of real numbers for which x, y, and z are not all zero. Also, (x_1, y_1, z_1) and (x_2, y_2, z_2) are regarded as equivalent (the same point) if one triple is a nonzero multiple of the other.

LINES: Ordered triples (a, b, c) of real numbers for which a, b, and c are not all zero. Also, (a_1, b_1, c_1) and (a_2, b_2, c_2) are regarded as equivalent (the same line) if one triple is a nonzero multiple of the other.

A POINT (x, y, z) is regarded as being on a LINE (a, b, c) if $ax + by + cz = 0$.

2.2.12 The Second Vector Plane

(Note: This is not a standard name.)

POINTS: Ordered triples (x, y, z) of real numbers for which z is nonzero. Also, (x_1, y_1, z_1) and (x_2, y_2, z_2) are regarded as equivalent (the same point) if one triple is a nonzero multiple of the other.

LINES: Ordered triples (a, b, c) of real numbers for which a and b are not both zero. Also, (a_1, b_1, c_1) and (a_2, b_2, c_2) are regarded as equivalent (the same line) if one triple is a nonzero multiple of the other.

A POINT (x, y, z) is regarded as being on a LINE (a, b, c) if $ax + by + cz = 0$.

2.2.13 Analytical Euclidean Space: $\mathbf{E}^3$

POINTS: Ordered triples (x, y, z) of real numbers.

LINES: Sets of points of the form…

PLANES: Sets of points that satisfy an equation of the form $ax + by + cz + d = 0$, where a, b, c and d are real numbers; and further a, b and c are not all zero.

2.2.14 Analytical Euclidean 4-Space: $\mathbf{E}^4$

POINTS:

LINES:

PLANES:

2.2.15 Analytical Euclidean n-Space: $\mathbf{E}^n$

Here, assume n is an integer greater than 3.

POINTS:

LINES:

PLANES:

2.3 The Analytical Model

The analytic model $\mathbf{E}^2$ for planar Euclidean geometry assigns the following meanings to points and lines: A *point* is an ordered pair (x, y) of real numbers. A *line* is a set of points satisfying an equation of the form $ax + by = c$, where a and b are not both zero.

Problem 2.3.1 Why do we want to prohibit a and b from both being zero in the equation for lines? Is there any problem with c being zero?

We will say that this line is represented by an equation in *standard form.*

Note that if we multiply an equation in standard form by a nonzero constant, then we get another equation in standard form that represents exactly the same line. So the representation of the line is not unique.

Problem 2.3.2 For each of the following pairs of points, find the line containing them.

1. $(1, 3)$ and $(3, -8)$.
2. $(1, 3)$ and $(3, 3)$.
3. $(1, 3)$ and $(1, -8)$.

Problem 2.3.3

1. What is the *point-slope* form of a line? Why is this name used? Can every line be expressed in this form?
2. What is the *slope-intercept* form of a line? Why is this name used? Can every line be expressed in this form?

Problem 2.3.4 Prove that points and lines in $\mathbf{E}^2$ (as defined above) satisfy Axiom 2.1.1.

Note that in order to do the previous problem, you need to prove two things: (1) Given any two different points, there is at least one line containing both of them; and (2) Given any two different points, there is no more than one line containing both of them.

Here is another formula for an equation in standard form of a line containing two given points.

Theorem 2.3.5

An equation of the line containing different points (x_1, y_1) *and* (x_2, y_2) *is*

$$(y_1 - y_2)x + (x_2 - x_1)y = x_2y_1 - x_1y_2.$$

Perhaps we should call this the *point-point* form of the line! How could we discover this formula? 5

Problem 2.3.6 Verify that this is the equation of a line. Where do you use the assumption that the two points are different?

Problem 2.3.7 Verify that each of the two points (x_1, y_1) and (x_2, y_2) satisfies the equation.

The problem above shows that there is *at least* one line containing two given different point.

Problem 2.3.8 Derive this formula by trying to solve the following two equations simultaneously for a, b and c, assuming that a and b are not both zero:

$$ax_1 + by_1 = c$$
$$ax_2 + by_2 = c$$

Problem 2.3.9 Explain how you can conclude from the previous problem that Axiom 2.1.1 holds for $\mathbf{E}^2$.

Problem 2.3.10 Use the formula to solve Problem 2.3.2.

2.4 Determinants

Definition 2.4.1 The following are formulas for *determinants* of arrays or *matrices* of numbers. We won't say more about determinants right now, but just learn the formulas:

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = (aei + bfg + cdh) - (afh + bdi + ceg).$$

Two other equivalent formulas for 3×3 matrices are:

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b\det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c\det\begin{bmatrix} d & e \\ g & h \end{bmatrix}.$$

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - d\det\begin{bmatrix} b & c \\ h & i \end{bmatrix} + g\det\begin{bmatrix} b & c \\ e & f \end{bmatrix}.$$

$$\det \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & \ell \\ m & n & o & p \end{bmatrix} =$$

$$a \det \begin{bmatrix} f & g & h \\ j & k & \ell \\ n & o & p \end{bmatrix} - b \det \begin{bmatrix} e & g & h \\ i & j & \ell \\ m & o & p \end{bmatrix} + c \det \begin{bmatrix} e & f & h \\ i & j & \ell \\ m & n & p \end{bmatrix} - d \det \begin{bmatrix} e & f & g \\ i & j & k \\ m & n & o \end{bmatrix}.$$

Another equivalent formula is:

$$\det \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & \ell \\ m & n & o & p \end{bmatrix} =$$

$$a \det \begin{bmatrix} f & g & h \\ j & k & \ell \\ n & o & p \end{bmatrix} - e \det \begin{bmatrix} b & c & d \\ j & k & \ell \\ n & o & p \end{bmatrix} + i \det \begin{bmatrix} b & c & d \\ f & g & h \\ n & o & p \end{bmatrix} - m \det \begin{bmatrix} b & c & d \\ f & g & h \\ j & k & \ell \end{bmatrix}.$$

Problem 2.4.2 Calculate the following determinants:

1. $$\det \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$$

2. $$\det \begin{bmatrix} 0 & 1 & 2 \\ -1 & 4 & 3 \\ -2 & 0 & 5 \end{bmatrix}$$

3.

$$\det \begin{bmatrix} -1 & 1 & 2 & -3 \\ 0 & -2 & 4 & 5 \\ 3 & 0 & 0 & -4 \\ 2 & 6 & 10 & -7 \end{bmatrix}$$

2.5 Equations of Lines via Determinants

Determinants can be used to express concisely the equation of a line determined by two points:

An equation of the line containing the distinct points (x_1, y_1) and (x_2, y_2) is

$$\det \begin{bmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ 1 & 1 & 1 \end{bmatrix} = 0.$$

Problem 2.5.1 Show that above statement is correct.

Problem 2.5.2 Use this formula to solve Problem 2.3.2.

2.6 Testing Collinearity

Theorem 2.6.1

Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear (are contained in a common line) if and only if

$$\det \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix} = 0.$$

Problem 2.6.2 Prove this statement. Suggestion: You might have to consider the special case that all three points are identical.

Problem 2.6.3 Use this formula to show that the points $A = (1, 2)$, $B = (1, 5)$ and $C = (2, -4)$ are not collinear.

Problem 2.6.4 Use this formula to show that the points $A = (1, 2)$, $B = (2, -4)$ and $C = (3, -10)$ are collinear.

Problem 2.6.5 If you have taken a course in matrix algebra, use what you have learned about matrices and independence of vectors to make sense of this theorem, thinking about the columns as vectors in three-dimensional space.

Problem 2.6.6 If a given triple of points is not collinear, then the determinant above is nonzero. What could be the geometric meaning of this number provided by the determinant? Try lots of examples and make a conjecture. Can you prove it?

2.7 Intersections of Lines

From a previous theorem we know that if two different lines intersect, then they intersect in exactly one point. Given two different lines in $\mathbf{E}^2$, how can we compute the coordinates of that point? One method is to use *Cramer's Rule*:

Theorem 2.7.1

If two different lines $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ intersect, then their point of intersection is given by:

$$x = \frac{\det \begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix}}{\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}}, \qquad y = \frac{\det \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}}{\det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}}.$$

Problem 2.7.2 Prove the above statement. Suggestion: Use matrix multiplication.

Reminder on multiplying matrices. If A and B are matrices, with A having the same number of columns as A has rows, then you can compute AB. (A special case of this is when A is an $\ell \times m$ matrix, and B is an $m \times 1$ matrix.) If A is $\ell \times m$ and B is $m \times n$ then $C = AB$ is $\ell \times n$. The entry in row i, column j of C will be the *inner product* or *dot product* of row i of A and column j of B. Here is a mnemonic to help you remember this, illustrated with an example. To visualize the calculation:

$$\begin{bmatrix} 0 & 6 & -1 \\ 2 & -4 & 7 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} -23 & 24 \\ 49 & -20 \end{bmatrix}$$

Arrange them this way:

			1	−2
			−3	4
			5	0
0	6	−1	−23	24
2	−4	7	49	−20

This theorem implies that if you have representations of two lines, and the two representations are not positive multiples of each other, then they cannot have more than one point in common, and hence cannot be the same line. This result, together with Problems 2.3.6 and 2.3.7, shows that there is one and only one line containing two given different points, and so confirms that the points and line of the analytic model satisfy Axiom 2.1.1.

Problem 2.7.3 What happens when you try to apply this formula to two lines that do not intersect, or to two equations describing the same line?

Problem 2.7.4 Practice using this formula with some examples of your own.

Problem 2.7.5 Try to make sense of the statement of the following theorem.

Theorem 2.7.6

If two different lines $a_1x + b_1y + c_1 = 0$ *and* $a_2x + b_2y + c_2 = 0$ *intersect, then their point of intersection is given by:*

$$\det \begin{bmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = 0.$$

Problem 2.7.7 Practice using this formula with some examples of your own.

Definition 2.7.8 Two or more lines are *concurrent* if they share a common point.

Theorem 2.7.9

The three lines given by $a_1x+b_1y+c_1=0$*,* $a_2x+b_2y+c_2=0$*, and* $a_3x+b_3y+c_3=0$ *are concurrent if and only if*

$$\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = 0.$$

Problem 2.7.10 Prove this theorem.

Problem 2.7.11 Practice using this formula with some examples of your own.

2.8 Parametric Equations of Lines

Here is another useful description of a line determined by two points, said to be in *parametric form.*

Theorem 2.8.1

> *If (x_1, y_1) and (x_2, y_2) are two distinct points, then the line containing them is the set of points $\{(x_1, y_1) + t(u, v) : t \in \mathbf{R}\}$, where $u = x_2 - x_1$ and $v = y_2 - y_1$.*

We will call (u, v) a direction vector *for the line.*

For example, if we have the points $(-2, 1)$ and $(1, 5)$, then the parametric equation of the line containing them is $(-2, 1) + t(3, 4)$.

Problem 2.8.2 Prove that the description given in the above theorem is correct; i.e., prove that this set is exactly the same as the set of points on the line containing the original two points, as given by the earlier formula in Theorem 2.3.5.

Thus any line can be expressed as a set of points of the form $\{(x_1, y_1) + t(u, v) : p \in \mathbf{E}\}$, u and v not both zero. It is helpful to think of (u, v) as a *vector,* specifying a particular change in x and y values. Parametric equations of lines are especially useful when describing lines in $\mathbf{E}^3$ (and higher dimensions!), and also lend themselves to computations for animations.

Note that it is easy to convert a line represented in slope-intercept form into a representation in parametric form. For example, if the line is given by $y = 3x - 7$, then let $x = t$:

$$\begin{aligned} (x, y) &= (x, 3x - 7) \\ &= (t, 3t - 7) \\ &= (0, -7) + t(1, 3). \end{aligned}$$

Problem 2.8.3 What point on the line do you get when $t = 0$? When $t = 1$? When $t = 1/2$? When $t = 1/3$? When $t = 2/3$? When $t = 2$? When $t = -1$? When $t = -1/2$? When $t = -4/3$? Try plotting these points and explain their geometric relationship to the original two points.

Problem 2.8.4 Use the formula to solve Problem 2.3.2.

Problem 2.8.5 If you have taken a course in matrix algebra, use what you have learned about Gaussian elimination applied to solving one equation in two variables to see the connection with obtaining a parametric equation of a line.

3 Coordinates and Distance

3.1 The SMSG Postulates and Theorems

The SMSG axioms are not independent, but they are very convenient for more quickly developing Euclidean geometry. Here are the postulates and associated definitions and theorems for the concepts of coordinates on a line, and distance, in the Euclidean plane.

1. Postulate 2. (The Distance Postulate.) To every pair of different points there corresponds a unique positive number.
2. Definition. The *distance* between two points is the positive number given by the Distance Postulate. If the points are P and Q, then the distance is denoted by PQ.
3. Postulate 3. (The Ruler Postulate.) The points of a line can be placed in correspondence with the real numbers in such a way that
   (a) To every point of the line there corresponds exactly one real number,
   (b) To every real number there corresponds exactly one point of the line, and
   (c) The distance between two points is the absolute value of the difference of the corresponding numbers.
4. Definition. A correspondence of the sort described in Postulate 3 is called a *coordinate system* for the line. The number corresponding to a given point is called the *coordinate* of the point.
5. Postulate 4. (The Ruler Placement Postulate.) Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive.
6. Definition. B is *between* A and C if (1) A, B and C are distinct points on the same line and (2) $AB + BC = AC$.
7. Theorem 2-1. Let A, B, C be three points of a line, with coordinates x, y, z. If $x < y < z$, then B is between A and C.
8. Theorem 2-2. Of any three different points on the same line, one is between the other two.

9. Theorem 2-3. Of three different points on the same line, only one is between the other two.

10. Definitions. For any two points A and B the *segment* $\overline{AB}$ is the set whose points are A and B, together with all points that are between A and B. The points A and B are called the *end-points* of $\overline{AB}$.

11. Definition. The distance AB is called the *length* of the segment $\overline{AB}$.

12. Definition. Let A and B be points of a line L. The *ray* $\overrightarrow{AB}$ is the set which is the union of (1) the segment $\overline{AB}$ and (2) the set of all points C for which it is true that B is between A and C. The point A is called the *end-point* of $\overrightarrow{AB}$.

13. Definition. If A is between B and C, then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are called *opposite rays.*

14. Theorem 2-4. (The Point Plotting Theorem.) Let $\overrightarrow{AB}$ be a ray, and let x be a positive number. Then there is exactly one point P of $\overrightarrow{AB}$ such that $AP = x$.

15. Definition. A point B is called a *midpoint* of a segment $\overline{AC}$ if B is between A and C, and $AB = BC$.

16. Theorem 2-5. Every segment has exactly one midpoint.

17. Definition. The midpoint of a segment is said to *bisect* the segment. More generally, any figure whose intersection with a segment is the midpoint of the segment is said to *bisect* the segment.

Note that *definitions* are not axioms or undefined terms. Rather, they can be thought of as *convenient collections of conditions,* making it easier to use the term, say, *line segment,* rather than constantly repeating the array of conditions that designate a set of points as a line segment every time we want to talk about one.

Problem 3.1.1 Using only the SMSG postulates above, together with the earlier Axiom 2.1.1 and Theorem 2.1.2 if needed, prove the above theorems.

3.2 The Analytic Model $\mathbf{E}^2$

We can show that the analytic model $\mathbf{E}^2$ satisfies the above postulates, once we make a definition.

Definition 3.2.1 The *distance* $d(P, Q)$ between two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is given by

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Problem 3.2.2 Explain how this definition of distance, together with the the parametric form of lines in the analytic model $\mathbf{E}^2$ enables us to confirm that SMSG Postulates 2–4 are satisfied by the analytic model. Suggestion: First rescale the vector (u, v) in the parametric form of a line by dividing it by $\sqrt{u^2 + v^2}$. The resulting vector (u', v') will then be a *unit vector;* i.e., will have length 1.

For example, if you begin with the parametric equation $(-2, 1) + t(3, 4)$, we compute $\sqrt{3^2 + 4^2} = 5$. Dividing $(3, 4)$ through by this number, we have a new parametric equation for the same line, $(-2, 1) + t(\frac{3}{5}, \frac{4}{5})$, where now the direction vector $(\frac{3}{5}, \frac{4}{5})$ has length 1.

Problem 3.2.3 Derive the midpoint formula for points $A = (x_1, y_1)$ and $B = (x_2, y_2)$.

$$\text{midpoint of } \overline{AB} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Problem 3.2.4 Assume we know that the Pythagorean Theorem holds in $\mathbf{E}^2$. Consider a third point $C = (x_1, y_2)$ to derive the formula for the distance between the points $A = (x_1, y_1)$ and $B = (x_2, y_2)$.

Problem 3.2.5 Assume we know that two lines L_1 and L_2 with respective direction vectors (u_1, v_1) and (u_2, v_2) are perpendicular if and only if (u_2, v_2) is a nonzero multiple of $(v_1, -u_1)$. Consider any right triangle ΔABC with right angle at A. Then there is a direction vector (u, v) and numbers s and t such that $B = A + s(u, v)$ and $C = A + t(v, -u)$. Use this, together with the distance formula, to prove that the Pythagorean Theorem holds.

3.3 Explorations on Distance

Use your general mathematical knowledge to think about the problems in this section.

Problem 3.3.1 What is the distance between two points in a (physical) field?

Problem 3.3.2 What is the distance between two locations in town? Does your answer change if there are any one-way streets? Does your answer change if you are walking, riding a bicycle, or driving a car?

Problem 3.3.3 What is the distance between two cities in the state?

Problem 3.3.4 What is the distance between two cities on the earth?

Problem 3.3.5 What is the distance traveled by a thrown rock? What is the distance along a curve in the shape of the St. Louis arch?

Problem 3.3.6 Explain what the arclength formula in calculus has to do with the formula for the distance between two points. What happens when you apply the calculus arclength formula to a portion of a linear function between two given points? Calculate the length of a segment of a catenary curve, given by

$$y = \frac{a}{2}(e^{\frac{x}{a}} + e^{\frac{-x}{a}}),$$

between $x = x_1$ and $x = x_2$.

Problem 3.3.7 What is the distance from the earth to the moon?

Problem 3.3.8 What is the distance between a speaker mounted on a wall of a room and the stereo system on the opposite wall?

Problem 3.3.9 What is the distance between two computers on the internet?

Problem 3.3.10 What does distance have to do with error-correcting codes?

Problem 3.3.11 Given a graph (network) with weights on the edges (e.g., this might represent a road network with distances between towns).

1. What is the most efficient way to find the shortest route between two given nodes?
2. What is the most efficient way to find a minimum weight subset of the edges that is a connected subgraph containing all the nodes?

Problem 3.3.12 Think about common (and uncommon!) notions of distance. What properties do we expect something called "distance" to satisfy?

Problem 3.3.13 Given two points, find the set of all points equidistant from both of them.

Problem 3.3.14 Given three points, find the set of all points equidistant from all three of them.

Problem 3.3.15 Given a finite set of points ("schools"), divide the plane up into regions ("school districts") according to which school is closest.

Problem 3.3.16 Given three points, find a point so that the sum of the distances to the three points is minimized.

Problem 3.3.17 Given an angle formed by two rays, find the set of all points equidistant from both rays.

Problem 3.3.18 Given a triangle, find the set of all points equidistant from all three sides.

Problem 3.3.19 Given a point and a line, find the set of all points equidistant from both of them.

Problem 3.3.20 Given two points, find the set of all points so that the sum of the distances to the two given points is a given constant c.

Problem 3.3.21 Given three points, find the shortest way to "connect them up." You may need to insert more points.

Problem 3.3.22 Given four points, find the shortest way to "connect them up." Try starting first with the four corners of a square.

Problem 3.3.23 A camper finds herself near (but not at) the bank of a straight river. Describe how to construct the shortest path from her current location to her tent, given that she wishes first to stop by the river.

If the river bank is represented by the line $y = 0$, her present location by the point $A = (0, 2)$, and her campsite by the point $B = (6, 3)$, what is the shortest route she can take? Provide justification. Make a good sketch. It may be helpful to use GeoGebra to experiment.

Problem 3.3.24 A camper finds herself at a point A near (but not at) the bank of a straight river. She can run at speed v and swim at speed w. She wants to get to a particular point B on the opposite bank of the river. So she runs to a point C on the near river bank and then swims from C to B. The water in the river is moving so slowly that during her swim you can neglect any movement downstream due to river flow. How can you determine the location of the point C?

Problem 3.3.25 A camper finds herself in the angle formed by the edge of a meadow and the bank of a river. Her tent is also in this angle. Describe how to construct the shortest path from her current location to her tent, given that she wishes to stop by the river on the way. Now describe how to construct the shortest path from her current location to her tent, given that she wishes first to stop by the river, and then after that stop by the meadow, on the way to her tent.

Try this specific example: The bank of the river is given by line $y = 0$. The edge of the meadow is given by the line $y = x$. The camper is currently at the point $(9, 6)$, and the tent is at the point $(6, 3)$. What is the shortest path from her current location to the river to the meadow to the tent?

Problem 3.3.26 Consider the set of all points $P(x, y)$ such that $x^2 - 2x + y^2 - 4y - 4 = 0$. What shape is this set? Provide justification. Why does this make sense? Find a better form of the equation that more clearly represents this set.

Problem 3.3.27 Let L be the line defined by the equation $y = 1$, and let $A = (4, 3)$. Consider the set of all points $P = (x, y)$ such that the distance from P to L equals the distance from P to A. Find an equation to describe this set of points, simplifying it as much as possible. Then use GeoGebra or a similar program to make a good sketch. What kind of shape do you get?

Problem 3.3.28 Let $A = (-2, 0)$ and $B = (2, 0)$. Consider the set of all points $P = (x, y)$ such that the sum of the distances $PA + PB$ equals 6. Find an equation to describe this set of points, simplifying it as much as possible—in particular, figure out how to get rid of any square roots. Then use GeoGebra or a similar program to make a good sketch. What kind of shape do you get?

Problem 3.3.29 Let $A = (-3, 0)$ and $B = (3, 0)$. Consider the set of all points $P = (x, y)$ such that the difference of the distances $|PA - PB|$ equals 2. Find an equation to describe this set of points, simplifying it as much as possible—in particular, figure out how to get rid of any square roots. Then use GeoGebra or a similar program to make a good sketch. What kind of shape do you get?

Problem 3.3.30 Explore the consequences of defining the distance AB between the points $A = (x_1, y_1)$ and (x_2, y_2) in $\mathbf{E}^2$ to be

$$AB = |(x_2 - x_1)| + |(y_2 - y_1)|.$$

Problem 3.3.31 Explore the consequences of defining the distance AB between the points $A = (x_1, y_1)$ and (x_2, y_2) in $\mathbf{E}^2$ to be

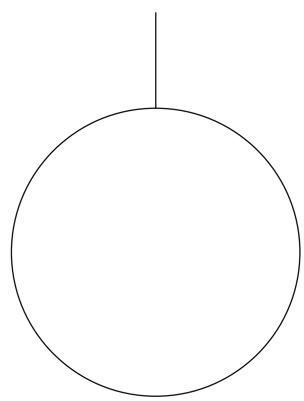
$$AB = \max\{|(x_2 - x_1)|, |(y_2 - y_1)|\}.$$

3.4 What is the Distance to the Horizon?

> *It was the first time that Poole had seen a genuine horizon since he had come to Star City, and it was not quite as far away as he had expected.... He used to be good at mental arithmetic—a rare achievement even in his time, and probably much rarer now. The formula to give the horizon distance was a simple one: the square root of twice your height times the radius—the sort of thing you never forgot, even if you wanted to...*
>
> —Arthur C. Clarke, *3001*, Ballantine Books, New York, 1997, page 71

Problem 3.4.1 In the above passage, Frank Poole uses a formula to determine the distance to the horizon given his height above the ground.

1. Use algebraic notation to express the formula Poole is using.
2. Beginning with the diagram below, derive your own formula. You will need to add some more elements to the diagram.



3. Compare your formula to Poole's; you will find that they do not match. How are they different?
4. When I was a boy it was possible to see the Atlantic Ocean from the peak of Mt. Washington in New Hampshire. This mountain is 6288 feet high. How far away is the horizon? Express your answer in miles. Assume that the radius of the Earth is 4000 miles. Use both your formula and Poole's formula and comment on the results. Why does Poole's formula work so well, even though it is not correct?

3.5 The Snowflake Curve

Begin with an equilateral triangle. Let's assume that each side of the triangle has length one. Remove the middle third of each line segment and replace it with two sides of an "outward-pointing" equilateral triangle of side length 1/3. Now you have a six-pointed star formed from 12 line segments of length 1/3. Replace the middle third of each of these line segments with two sides of outward equilateral triangle of side length 1/9. Now you have a star-shaped figure with 48 sides. Continue to repeat this process, and the figure will converge to the "Snowflake Curve." Shown below are the first three stages in the construction of the Snowflake Curve.

Problem 3.5.1

1. In the limit, what is the length of the Snowflake Curve?
2. In the limit, what is the area enclosed by the Snowflake Curve?

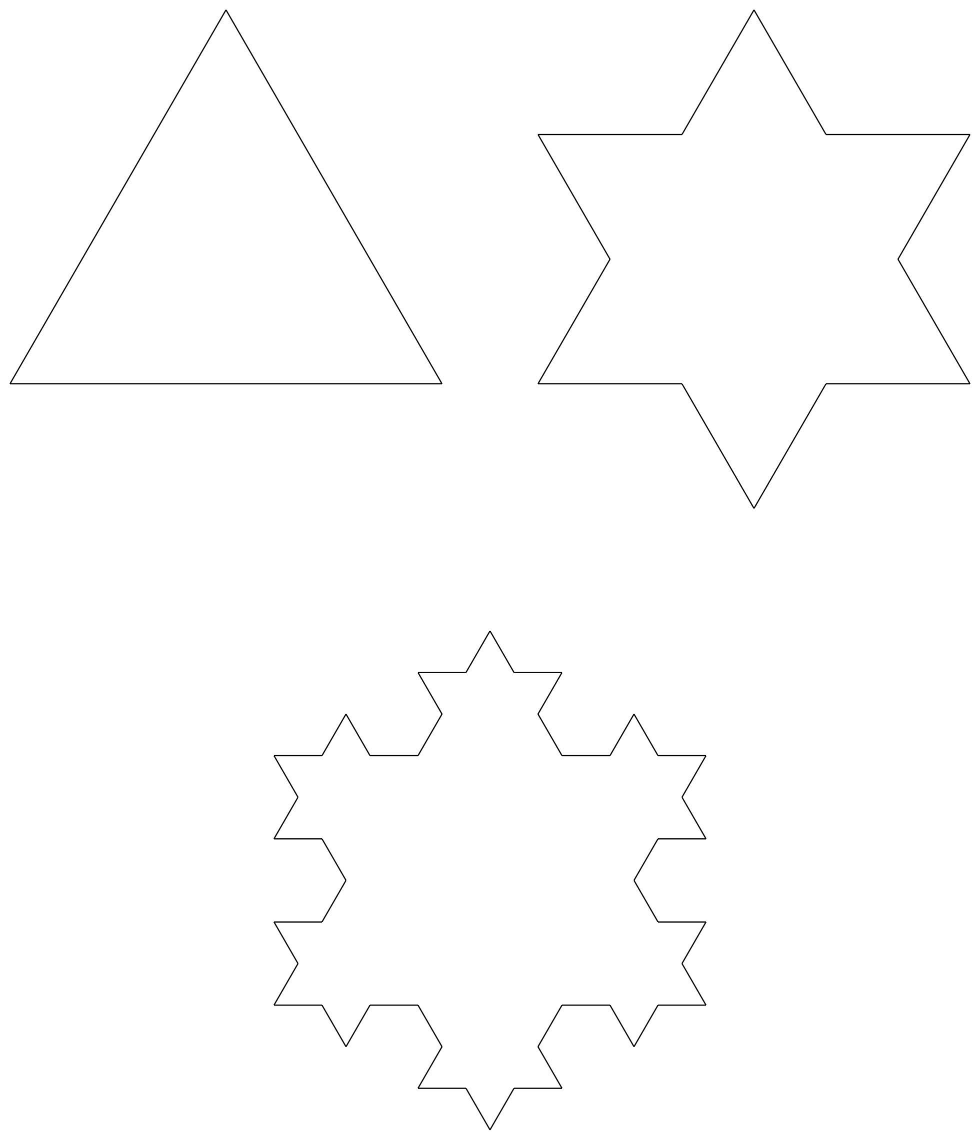


Figure 1: Constructing the Snowflake Curve

3.6 The Longimeter

How can we measure the lengths of curves in "real life?" There are devices consisting of wheels with some sort of dial that you can roll over a map to estimate distances, and larger versions that you can roll in front of you on, e.g., paths, to measure distance (what are these things called?). You can also estimate the distance that you walk by wearing a pedometer.

Here is another way to estimate the length of a curve on a map, using a simple device called a *longimeter*. On a transparent sheet of plastic create a square grid, each square having side length of, say 1 mm. Superimpose this grid your curve in three different orientations, differing one from the other by a rotation of 30°. In each of the three cases, count how many squares the curve passes through. Let the sum of these three numbers be S. Then an estimate of the length of the curve is $S/3.82$ mm.

In the example below, I rotated the figure rather than the grid. Each square has side length 0.25 in. The sum S is $16 + 16 + 15 = 47$, so the estimate of the length of the curve is $47/3.82 \approx 12.30$ units of length 0.25 in, or 3.07 in.

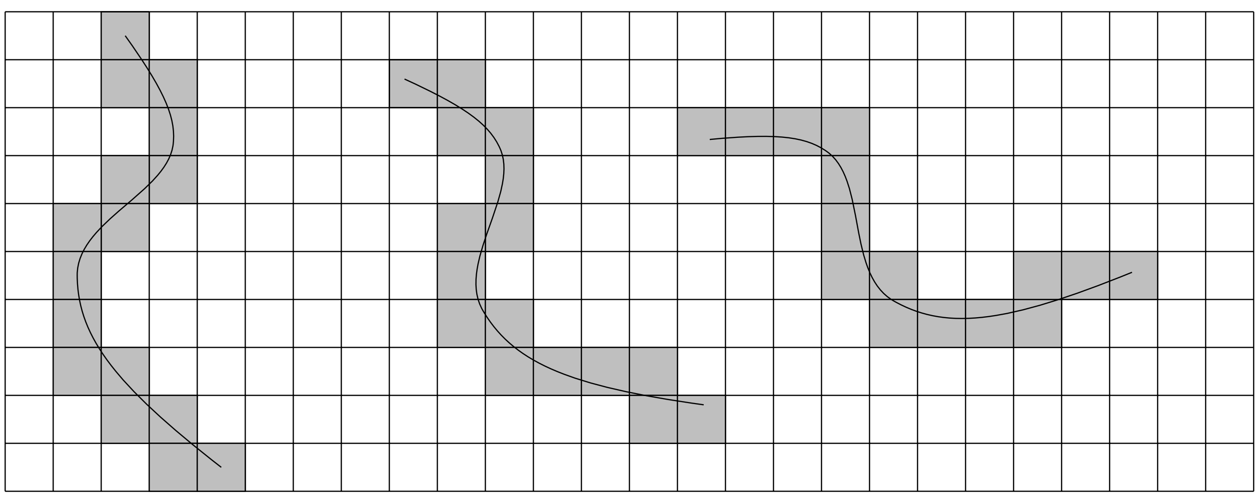


Figure 2: Using a Longimeter

Problem 3.6.1 Read the reference below and write up an explanation of why this method works. In particular, where does the number 3.82 come from?

Reference: H. Steinhaus, *Mathematical Snapshots*, Oxford University Press, New York, 1989, pp. 105–107.

3.7 Fractals

The notion of "length" of certain naturally occurring objects can, however, be tricky, and can lead one into the notion of fractals. The following quote comes from a book by Mandelbrot:

> *To introduce a first category of fractals, namely curves whose fractal dimension is greater than* 1*, consider a stretch of coastline. It is evident that its length is at least equal to the distance measured along a straight line between its beginning and its end. However, the typical coastline is irregular and winding, and there is no question it is much longer than the straight line between its end points.*
>
> *There are various ways of evaluating its length more accurately... The result is most peculiar: coastline length turns out to be an elusive notion that slips between the fingers of one who wants to grasp it. All measurement methods ultimately lead to the conclusion that the typical coastline's length is very large and so ill determined that it is best considered infinite....*
>
> *Set dividers to a prescribed opening ϵ, to be called the yardstick length, and walk these dividers along the coastline, each new step starting where the previous step leaves off. The number of steps multiplied by ϵ is an approximate length $L(\epsilon)$. As the dividers' opening becomes smaller and smaller, and as we repeat the operation, we have been taught to expect $L(\epsilon)$ to settle rapidly to a well-defined value called the* true length*. But in fact what we expect does not happen. In the typical case, the observed $L(\epsilon)$ tends to increase without limit.*
>
> *The reason for this behavior is obvious: When a bay or peninsula noticed on a map scaled to* $1/100,000$ *is reexamined on a map at* $1/10,000$*, subbays and subpeninsulas become visible. On a* $1/1,000$ *scale map, sub-subbays and sub-subpeninsulas appear, and so forth. Each adds to the measured length.*
>
> —B.B. Mandelbrot, "How Long is the Coast of Britain," *The Fractal Geometry of Nature*, W.H. Freeman and Company, New York, 1983, Chapter 5, p. 25.

3.8 The Triangle Inequality in $\mathbf{E}^2$

Problem 3.8.1 For two ordered pairs $A = (x_1, y_1)$ and $B = (x_2, y_2)$, define

$$A \cdot B = x_1 x_2 + y_1 y_2$$

For an ordered pair $A = (x_1, y_1)$, define

$$\|A\| = \sqrt{x_1^2 + y_1^2} = \sqrt{A \cdot A}$$

Prove the following theorem directly from the definitions:

$$A \cdot B \leq \|A\| \|B\|$$

Problem 3.8.2 Prove:

$$(A + B) \cdot (A + B) = \|A\|^2 + 2A \cdot B + \|B\|^2$$

Problem 3.8.3 Observe the obvious fact that

$$AB = \|B - A\| = \sqrt{(B - A) \cdot (B - A)}$$

Prove the *Triangle Inequality* holds for any three points A, B, C:

$$AC \leq AB + BC$$

Suggestion: First prove that

$$\|D + E\| \leq \|D\| + \|E\|$$

Then let $D = B - A$ and $E = C - B$.

Problem 3.8.4 Check that all of the problems in this section can be generalized to $\mathbf{E}^3$ as well.

4 Lines and Planes in Space

4.1 The SMSG Postulates and Theorems

1. Definition. The set of all points is called *space.*
2. Definition. A set of points is *collinear* if there is a line which contains all the points of the set.
3. Definition. A set of points is *coplanar* if there is a plane which contains all the points of the set.
4. Postulate 5.
   (a) Every plane contains at least three non-collinear points.
   (b) Space contains at least four non-coplanar points.
5. Theorem 3-1. Two different lines intersect in at most one point.
6. Postulate 6. If two points lie in a plane, then the line containing these points lies in the same plane.
7. Theorem 3-2. If a line intersects a plane not containing it, then the intersection is a single point.
8. Postulate 7. Any three points lie in at least one plane, and any three non-collinear points lie in exactly one plane. More briefly, any three points are coplanar, and any three non-collinear points determine a plane.
9. Theorem 3-3. Given a line and a point not on the line, there is exactly one plane containing both of them.
10. Theorem 3-4. Given two intersecting lines, there is exactly one plane containing them.
11. Postulate 8. If two different planes intersect, then their intersection is a line.

Problem 4.1.1 Using only the set of SMSG postulates and theorems provided so far, prove the above theorems.

4.2 The Analytic Model $\mathbf{E}^3$

The analytic model for points, lines, and planes in space requires some definitions to assign meanings to our terms.

Definition 4.2.1

1. A *point* is an ordered triple (x, y, z) of real numbers.
2. A *line* is a set of points of the form $\{(x_1, y_1, z_1) + t(u, v, w)\}$ where at least one of u, v, w is not zero. (Note that this representation is not unique.) The vector (u, v, w) is called a *direction vector* of the line.
3. A *plane* is a set of points of the form $\{(x, y, z) : ax + by + cz = d\}$ where at least one of a, b, c is not zero. (Note that you can multiply the equation of a plane by a nonzero constant to get another equation of the same plane.)
4. The *distance* $d(P, Q)$ between two points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ is given by

$$d((x_1, y_1, z_1), (x_2, y_2, z_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Problem 4.2.2 Why is it important that at least one of a, b, c is not zero in the equation of a plane?

Problem 4.2.3 Prove that, with these definitions, the analytic model satisfies SMSG Postulates 5–8.

Theorem 4.2.4 *An equation of a plane containing three non-collinear points* (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) *is given by*

$$\det \begin{bmatrix} x & x_1 & x_2 & x_3 \\ y & y_1 & y_2 & y_3 \\ z & z_1 & z_2 & z_3 \\ 1 & 1 & 1 & 1 \end{bmatrix} = 0.$$

Problem 4.2.5 Prove the above theorem. Practice with some examples. What happens when you try to use this equation when the three points are collinear?

Theorem 4.2.6 *Four points* (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), *and* (x_4, y_4, z_4) *are coplanar if and only if*

$$\det \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} = 0.$$

Problem 4.2.7 Prove the above theorem. Practice with some examples.

Problem 4.2.8 How can you determine the point of intersection of a line and a plane? Practice with some examples. What happens with your procedure if the line does not intersect the plane?

Problem 4.2.9 If you are familiar with a matrix algebra, explain how Gaussian elimination enables you to find the line that is the intersection of two different intersecting planes. Practice with some examples.

Problem 4.2.10 Make sense of the statement of the following theorem.

Theorem 4.2.11 If three planes $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$, $a_3x + b_3y + c_3z + d_3 = 0$ have exactly one point in common, then

$$\det \begin{bmatrix} a & b & c & d \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} = 0$$

Theorem 4.2.12 *Four planes $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$, $a_3x + b_3y + c_3z + d_3 = 0$, and $a_4x + b_4y + c_4z + d_4 = 0$ are concurrent (share a common point) if and only if*

$$\det \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} = 0$$

5 Convex Sets

5.1 The SMSG Postulates and Theorems

1. Definition. A set A is called *convex* if for every two points P and Q or A, the entire segment $\overline{PQ}$ lines in A.

2. Postulate 9. (The Plane Separation Postulate.) Given a line and a plane containing it. The points of the plane that do not lie on the line form two sets such that (1) each of the sets is convex and (2) if P is in one set and Q is in the other then the segment $\overline{PQ}$ intersects the line.

3. Definitions. Given a line L and a plane E containing it, the two sets determined by Postulate 9 are called *half-planes,* and L is called an *edge* of each of them. We say that L *separates* E into the two half-planes. If two points P and Q or E lie in the same half-plane, we say that they lie *on the same side* of L; if P lies in one of the half-planes and Q in the other they lie *on opposite sides* of L.

4. Postulate 10. (The Space Separation Postulate.) The points of space that do not lie in a given plane form two sets such that (1) each of the sets is convex and (2) if P is one set and Q is in the other, then the segment $\overline{PQ}$ intersects the plane.

5. Definitions. The two sets determined by Postulate 10 are called *half-spaces,* and the given plane is called the *face* of each of them.

5.2 The Analytic Models $\mathbf{E}^2$ and $\mathbf{E}^3$

Problem 5.2.1 Assume that $\{(x, y) : ax + by = c\}$ is a line in $\mathbf{E}^2$. Prove that that the two sets $\{(x, y) : ax + by < c\}$ and $\{(x, y) : ax + by > c\}$ satisfy SMSG Postulate 9; i.e., that together they contain all of the points that do not lie on the line, that each of the sets is convex, and if P is in one set and Q is in the other then the segment $\overline{PQ}$ intersects the line.

Problem 5.2.2 Assume that $\{(x, y, z) : ax + by + cz = d\}$ is a plane in $\mathbf{E}^3$. Prove that that the two sets $\{(x, y, z) : ax + by + cz < d\}$ and $\{(x, y, z) : ax + by + cz > d\}$ satisfy SMSG Postulate 10; i.e., that together they contain all of the points that do not lie on the plane, that each of the sets is convex, and if P is in one set and Q is in the other then the segment $\overline{PQ}$ intersects the plane.

6 Angles

6.1 The SMSG Postulates and Theorems

1. Definitions. An *angle* is the union of two rays which have the same end-point but do not lie in the same line. The two rays are called the *sides* of the angle, and their common end-point is called the *vertex.*

2. Notation. The angle which is the union of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is denoted by $\angle BAC$, or by $\angle CAB$, or simply by $\angle A$ if it is clear which rays are meant.

3. Definitions. If A, B, and C are any three non-collinear points, then the union of the segments $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ is called a *triangle,* and is denoted by ΔABC; the points A, B and C are called its *vertices,* and the segments $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ are called its *sides.* Every triangle determines three angles; ΔABC determines the angles $\angle BAC$, $\angle ABC$ and $\angle ACB$, which are called the *angles of* ΔABC. For short, we will often write them simply as $\angle A$, $\angle B$, and $\angle C$.

4. Definitions. Let $\angle BAC$ be an angle lying in plane E. A point P of E lines in the *interior* of $\angle BAC$ if (1) P and B are on the same side of the line $\overleftrightarrow{AC}$ and also (2) P and C are on the same side of the line $\overleftrightarrow{AB}$. The *exterior* of $\angle BAC$ is the set of all points of E that do not line in the interior and do not lie on the angle itself.

5. Definitions. A point lies in the *interior* of a triangle if it lie in the interior of each of the angles of the triangle. A point lies in the *exterior* of a triangle if it lies in the plane of the triangle but is not a point of the triangle or of its interior.

6. Postulate 11. (The Angle Measurement Postulate.) To every angle $\angle BAC$ there corresponds a real number between 0 and 180.

7. Definition. The number specified by Postulate 11 is called the *measure of the angle,* and written as $\mathrm{m}\angle BAC$.

8. Postulate 12. (The Angle Construction Postulate.) Let $\overrightarrow{AB}$ be a ray on the edge of the half-plane H. For every number r between 0 and 180 there is exactly one ray $\overrightarrow{AP}$, with P in H, such that $\mathrm{m}\angle PAB = r$.

9. Postulate 13. (The Angle Addition Postulate.) If D is a point in the interior of $\angle BAC$, then $\mathrm{m}\angle BAC = \mathrm{m}\angle BAD + \mathrm{m}\angle DAC$.

10. Definition. If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays, and $\overrightarrow{AD}$ is another ray, then $\angle BAD$ and $\angle DAC$ form a *linear pair.*

11. Definition. If the sum of the measures of two angles is 180, then the angles are called *supplementary,* and each is called a *supplement* of the other.

12. Postulate 14. (The Supplement Postulate.) If two angles form a linear pair, then they are supplementary.

13. Definitions. If the two angles of a linear pair have the same measure, then each of the angles is a *right angle.*

14. Definition. Two intersecting sets, each of which is either a line, a ray or a segment, are *perpendicular* if the two lines which contain them determine a right angle.

15. Definition. If the sum of the measures of two angles is 90, then the angles are called *complementary,* and each of them is called a *complement* of the other.

16. Definition. An angle with measure less than 90 is called *acute,* and an angle with measure greater than 90 is called *obtuse.*

17. Definition. Angles with the same measure are called *congruent angles.*

18. Theorem 4-1. If two angles are complementary, then both of them are acute.

19. Theorem 4-2. Every angle is congruent to itself.

20. Theorem 4-3. Any two right angles are congruent.

21. Theorem 4-4. If two angles are both congruent and supplementary, then each of them is a right angle.

22. Theorem 4-5. Supplements of congruent angles are congruent.

23. Theorem 4-6. Complements of congruent angles are congruent.

24. Definition. Two angles are *vertical angles* if their sides form two pairs of opposite rays.

25. Theorem 4.7. Vertical angles are congruent.

26. Theorem 4-8. If two intersecting lines form one right angle, then they form four right angles.

Problem 6.1.1 Using only the set of SMSG postulates and theorems provided so far, prove the above theorems.

6.2 Radians

Suppose you have a circle of radius 1. Its circumference is $C = 2\pi r = 2\pi$, which is a bit bigger than 6.2.

Problem 6.2.1 Explain why the formula for the circumference of a circle provides the *definition* of π.

The measure of a central angle that cuts off a piece (intercepts an arc) of the circumference of length 1 is called a *radian*. In general, the measure of an angle that intercepts an arc of the circumference having length ℓ is said to have measure ℓ radians. Therefore, there are 2π radians around the center of a circle and we can convert back and forth between degrees and radians by

$$\theta(\text{in radians}) = \frac{\pi}{180^\circ}\theta(\text{in degrees})$$

$$\theta(\text{in degrees}) = \frac{180^\circ}{\pi}\theta(\text{in radians})$$

Using radians makes many formulas look "nicer." For example, Suppose C is a circle of radius r. The length ℓ of an arc intercepted by a central angle θ is given by

$$\ell = r\theta \quad (\text{if } \theta \text{ is measured in radians})$$

$$\ell = \frac{\pi}{180^\circ}r\theta \quad (\text{if } \theta \text{ is measured in degrees})$$

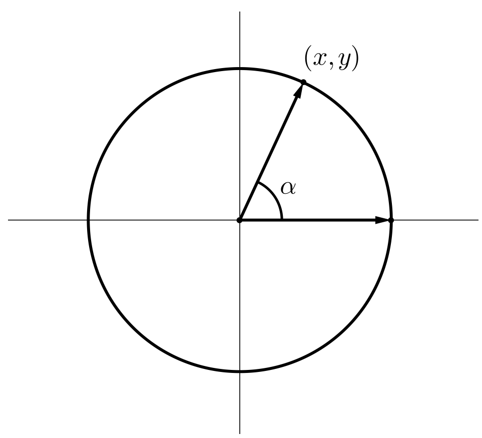
So the radian measure of the central angle is the ratio of the length of the arc and the radius.

Problem 6.2.2 Propose an analogous definition of the measure of a solid angle where three, four, or more planes meet at common vertex of a polyhedron, and explain why your definition is reasonable. Then look up the official name and definition of solid angle measure.

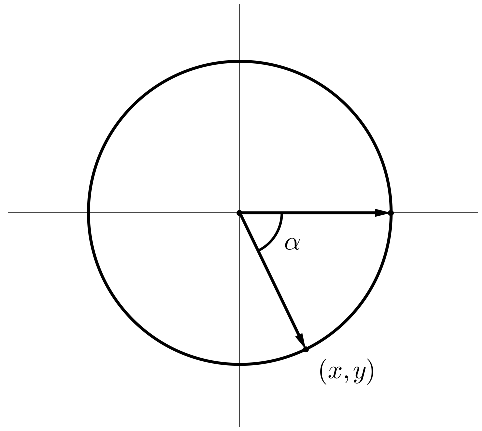
6.3 Trigonometric Functions in the Analytic Model $\mathbf{E}^2$

A circle of radius one is called a *unit circle*. A unit circle with center at the origin of the Cartesian plane is often called *the unit circle*. The trigonometric functions sine, cosine, tangent, secant, cosecant, and cotangent, can be defined using the unit circle.

Let α be the radian measure of an angle. Place a ray r from the origin along the x axis. If $\alpha \geq 0$, rotate the ray by α radians in the counterclockwise direction.



If $\alpha < 0$, rotate the ray by $|\alpha|$ radians in the clockwise direction.



Determine the point (x, y) where r intersects the unit circle. We define

$$\cos\alpha = x$$

and

$$\sin\alpha = y.$$

Define also

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}.$$
$$\sec\alpha = \frac{1}{\cos\alpha},$$
$$\csc\alpha = \frac{1}{\sin\alpha},$$
$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha}.$$

Problem 6.3.1 Use the definitions for the sine, cosine, and tangent functions to evaluate $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$ when α equals

1. 0
2. $\frac{\pi}{2}$
3. π

4. $\frac{3\pi}{2}$

5. 2π

6. $\frac{\pi}{3}$

7. $\frac{\pi}{4}$

8. $\frac{\pi}{6}$

9. $\frac{n\pi}{3}$ for all possible integer values of n

10. $\frac{n\pi}{4}$ for all possible integer values of n

11. $\frac{n\pi}{6}$ for all possible integer values of n

Problem 6.3.2 Drawing on the definitions for the sine and cosine functions, sketch the graphs of the functions $f(\alpha) = \sin\alpha$ and $f(\alpha) = \cos\alpha$, and explain how you can deduce these naturally from the unit circle definition,

Problem 6.3.3 Continuing to think about the unit circle definition, complete the following formulas and give brief explanations, including a diagram, for each.

1. $\sin(-\alpha) = -\sin(\alpha)$.

2. $\cos(-\alpha) =$

3. $\sin(\pi + \alpha) =$

4. $\cos(\pi + \alpha) =$

5. $\sin(\pi - \alpha) =$

6. $\cos(\pi - \alpha) =$

7. $\sin(\pi/2 + \alpha) =$

8. $\cos(\pi/2 + \alpha) =$

9. $\sin(\pi/2 - \alpha) =$

10. $\cos(\pi/2 - \alpha) =$

11. $\sin^2(\alpha) + \cos^2(\alpha) =$

Problem 6.3.4 Use GeoGebra to make a sketch of the unit circle to illustrate what you have learned so far.

Problem 6.3.5 Use the sine and cosine functions to determine the coordinates of the vertices of the following. In each case except the last two, choose one vertex to be the point $(1, 0)$.

1. A regular triangle with vertices having a distance of 1 from the origin.
2. A regular square with vertices having a distance of 1 from the origin.
3. A regular pentagon with vertices having a distance of 1 from the origin.
4. A regular hexagon with vertices having a distance of 1 from the origin.
5. A regular heptagon with vertices having a distance of 3 from the origin.
6. A regular n-gon with vertices having a distance of r from the origin.

Problem 6.3.6 Confirm the above calculations by entering the coordinates of the above points into GeoGebra.

Problem 6.3.7 Here is perhaps a more familiar way to define sine and cosine for an acute angle α: Take any right triangle for which one of the angles measures α. Then $\sin\alpha$ is the ratio of the lengths of the opposite side and the hypotenuse, and $\cos\alpha$ is the ratio of the lengths of the adjacent side and the hypotenuse. Explain why this definition gives the same result as the unit circle.

Problem 6.3.8 Describe a procedure to determine the rectangular coordinates (x, y) of a point from its polar coordinates (r, θ) and justify why it works.

Problem 6.3.9 Look up the definitions of cylindrical and spherical coordinates.

1. Justify the following conversion from cylindrical coordinates (r, θ, z) to rectangular coordinates (x, y, z).

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

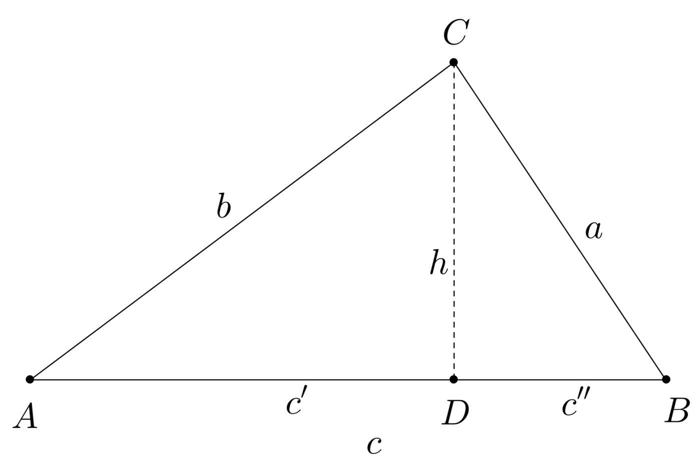
2. Justify the following conversion from spherical coordinates (r, θ, ϕ) to rectangular coordinates (x, y, z).

$$\begin{aligned} x &= r\cos\theta\sin\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\phi \end{aligned}$$

6.4 Trigonometric Identities

Problem 6.4.1 In this problem we will use the triangle pictured below. In this triangle all angles have measure less than 90°; however, the results hold true for general triangles.

The lengths of $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$ are a, b and c, respectively. Segment $\overline{AD}$ has length c' and $\overline{DB}$ length c''. Segment $\overline{CD}$ is the altitude of the triangle from C, and has length h.



The usual formula for the area of a triangle is $\frac{1}{2}$(base)(height), as you probably already know.

1. Prove that $\operatorname{area}(\triangle ABC) = \frac{1}{2}bc\sin A$.
2. What is a formula for $\operatorname{area}(\triangle ABC)$ using $\sin B$? Using $\sin C$? (Note: you will have to use the altitude from A or B).
3. What is the relationship of these formulas to the SAS triangle congruence criterion?

Problem 6.4.2 Using the same triangle, the *Law of Sines* is:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

1. We showed that the area of this triangle was given by three different formulas. What are they?
2. From these three formulas, prove the Law of Sines.

Problem 6.4.3 The *Law of Cosines* is:

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

Using the above triangle:

1. Show that $c' = b\cos A$.

2. Observe the obvious fact that $c'' = c - c'$.
3. Verify that $h^2 = b^2 - (c')^2$.
4. Apply the Pythagorean Theorem to triangle $\triangle CDB$, then use the facts above to make the appropriate substitutions to prove the Law of Cosines.
5. What happens when you apply the Law of Cosines in the case that $\angle A$ is a right angle?

Problem 6.4.4 Suppose for a triangle you are given the lengths of the three sides. How can you determine the measures of the three angles?

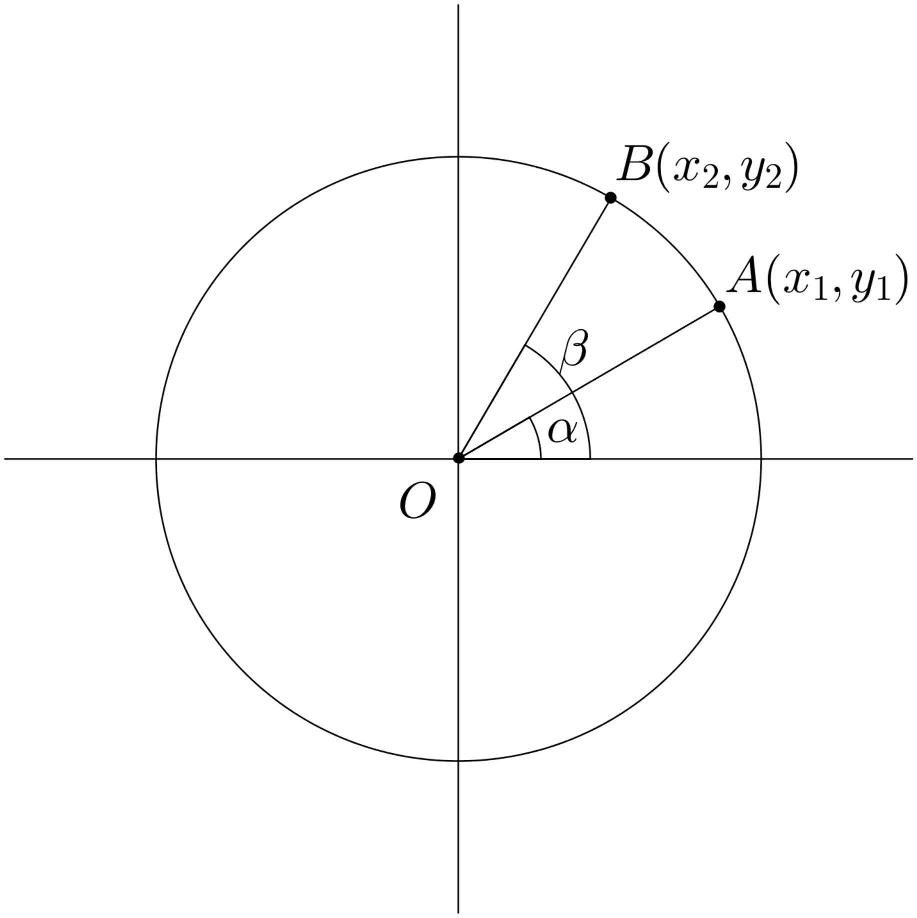
Problem 6.4.5 Suppose for a triangle you are given the lengths of two sides and the measure of the included angle. How can you determine the length of the other side, and the measures of the other two angles?

Problem 6.4.6 Suppose for a triangle you are given the measures of two angles and the length of the included side. How can you determine the measure of the other angle, and the lengths of the other two sides?

Problem 6.4.7 Assume that you have triangle $\triangle ABC$ such that the coordinates of the three (distinct) points A, B, and C are $(0, 0)$, (x_1, y_1), and (x_2, y_2), respectively. Use the Law of Cosines and the distance formula to prove that

$$\cos A = \frac{x_1x_2 + y_1y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}.$$

Problem 6.4.8 Assume that A and B are two points on the unit circle centered at the origin, with respective coordinates (x_1, y_1) and (x_2, y_2). Draw the line segments $\overline{OA}$ and $\overline{OB}$. Let α be the angle that $\overline{OA}$ makes with the positive x-axis, and β be the angle that $\overline{OB}$ makes with the positive x-axis.



1. From Problem 6.4.7 we know that

$$\cos(\angle AOB) = \frac{x_1x_2 + y_1y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}$$

From this, prove that

$$\cos(\beta - \alpha) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

2. Replace α with $-\alpha$ in the previous equation to prove

$$\cos(\beta + \alpha) = \cos\beta\cos\alpha - \sin\beta\sin\alpha.$$

3. Replace β with $\pi/2 - \gamma$ and α with $-\delta$ in the previous equation to prove

$$\sin(\gamma + \delta) = \sin\gamma\cos\delta + \cos\gamma\sin\delta.$$

4. Replace δ with $-\delta$ in the previous equation to prove

$$\sin(\gamma - \delta) = \sin\gamma\cos\delta - \cos\gamma\sin\delta.$$

The above four formulas are the trigonometric *angle sum* and *angle difference formulas.*

Problem 6.4.9 Prove the *double angle* formulas:

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha.$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha.$$

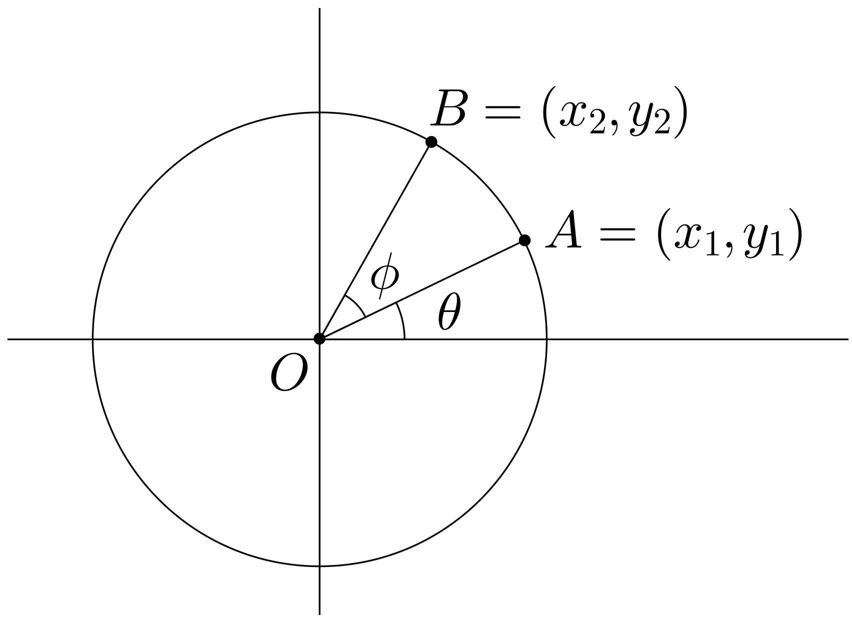
Problem 6.4.10 Prove the *half angle* formulas for angle $0 \leq \beta \leq \frac{\pi}{2}$.

$$\sin(\beta/2) = \sqrt{\frac{1-\cos\beta}{2}}.$$

$$\cos(\beta/2) = \sqrt{\frac{1+\cos\beta}{2}}.$$

6.5 Rotations

Problem 6.5.1 Now assume that we have a circle of radius r, that point A has coordinates $(x_1, y_1) = (r\cos\theta, r\sin\theta)$, and that we wish to rotate it by ϕ about the origin, obtaining the point $B = (x_2, y_2) = (r\cos(\theta+\phi), r\sin(\theta+\phi))$.



1. Prove that
$$(x_2, y_2) = (x_1\cos\phi - y_1\sin\phi, x_1\sin\phi + y_1\cos\phi).$$

2. Conclude that:

> The matrix for the rotation centered at the origin by the angle ϕ is
> $$\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$
> where $c = \cos\phi$ and $s = \sin\phi$.

That is to say, prove that

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

Problem 6.5.2 Find the matrices for the rotations about the origin by each angle ϕ, $0 \leq \phi < 360°$, that is a multiple of 90 degrees.

Problem 6.5.3 Find the matrices for the rotations about the origin by each angle ϕ, $0 \leq \phi < 360°$, that is a multiple of 45 degrees.

Problem 6.5.4 Find the matrices for the rotations about the origin by each angle ϕ, $0 \leq \phi < 360°$, that is a multiple of 30 degrees.

6.6 Complex Numbers

Any complex number $z = a + bi$ can be represented by a point (a, b) in the Cartesian plane. The real number a is called the *real part* of z, and the real number (not including the i) is called the *imaginary part* of z. But you can set $r = \sqrt{a^2 + b^2}$ and find θ such that $\cos\theta = a/r$ and $\sin\theta = b/r$. That is, (r, θ) are polar coordinates for the point (a, b). Then $z = r(\cos\theta + i\sin\theta)$. The angle θ is called the *argument* of z, denoted $\arg z$, and the length r is called the *modulus* of z, denoted $|z|$. Note: Sometimes $r(\cos\theta + i\sin\theta)$ is written $r\text{cis}\,\theta$.

Problem 6.6.1 Suppose $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, corresponding to the points $P_1 = (a_1, b_1)$ and $P_2 = (a_2, b_2)$, respectively, in the Cartesian plane. Explain how to find $z = z_1 + z_2$ geometrically. Explain how to find $z = z_1 - z_2$ geometrically.

Problem 6.6.2 Suppose $z_1 = r_1\text{cis}\,\theta_1$ and $z_2 = r_2\text{cis}\,\theta_2$, corresponding to the points P_1, P_2 in the Cartesian plane with polar coordinates (r_1, θ_1), (r_2, θ_2), respectively. Explain how to find $z = z_1 z_2$ geometrically. Explain how to find $z = z_1/z_2$ geometrically.

Problem 6.6.3 Prove the following:

> Let $w = r(\cos\phi + i\sin\phi)$. Then the function $f(z) = wz$ rotates the complex plane counterclockwise by the angle ϕ and then scales it by a factor of r.

Problem 6.6.4 From what you learned in the previous exercise,

1. Explain geometrically what multiplying by i does.
2. Show geometrically that $i^2 = -1$.
3. Find three complex numbers such that $z^3 = 1$.
4. Find three complex numbers such that $z^3 = 27$.
5. Find four complex numbers such that $z^4 = 1$.

6. Find four complex numbers such that $z^4 = \frac{1}{16}$.

7. Find six complex numbers such that $z^6 = 1$.

8. Find two complex numbers such that $z^2 = i$.

9. Find three complex numbers such that $z^3 = 8i$.

10. Explain how to calculate z^n for any particular complex number z, where n is a positive integer.

11. Explain how to find all solutions to any equation of the form $z^n = z_0$ where n is a positive integer and z_0 is a particular complex number.

Problem 6.6.5 Show that if we map or identify the complex number $x + iy = r\text{cis}\,\theta$ with the 2×2 matrix

$$\begin{bmatrix} rc & -rs \\ rs & rc \end{bmatrix}, \text{ equivalently, } \begin{bmatrix} x & -y \\ y & x \end{bmatrix},$$

where $c = \cos\theta$ and $s = \sin\theta$, then we can add and multiply complex numbers by simply adding and multiplying their associated matrices. Thus, this set of matrices is a *representation* of, or *isomorphic* to, the complex numbers.

Note also that the subset of matrices of the form

$$\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$$

is isomorphic to the set of real numbers.

We have seen that radians are a natural unit for getting a nice formula for the length of a circular arc: $\ell = r\theta$ if θ is the central angle measured in radians, r is the radius of the circle, and ℓ is the length of the arc. Another motivation for expressing angles is radians is the Taylor series formulas for sine and cosine:

For angle x measured in radians:

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \frac{1}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Problem 6.6.6 Derive these Taylor series.

Problem 6.6.7 Sum the squares of the above series to verify that $\sin^2 x + \cos^2 x = 1$.

This might remind you of the Taylor series for e^x:

$$e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$

Problem 6.6.8 Derive this Taylor series.

Problem 6.6.9 Use the above series to show that $e^a e^b = e^{a+b}$.

From substitution (and some observations about convergence), one gets the beautiful formula for all complex numbers x:

$$e^{ix} = \cos x + i \sin x$$

In particular, setting $x = \pi$ yields an expression containing the perhaps five most important constants in mathematics:

$$e^{i\pi} + 1 = 0$$

These formulas provide a connection between two representations for complex numbers on the one hand, and Cartesian and polar coordinates on the other. Any complex number $r\text{cis}\theta$ can now also be written $re^{i\theta}$. Because $r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$ we have another way to see that to multiply two complex numbers we add the angles (arguments) and multiply the lengths (moduli).

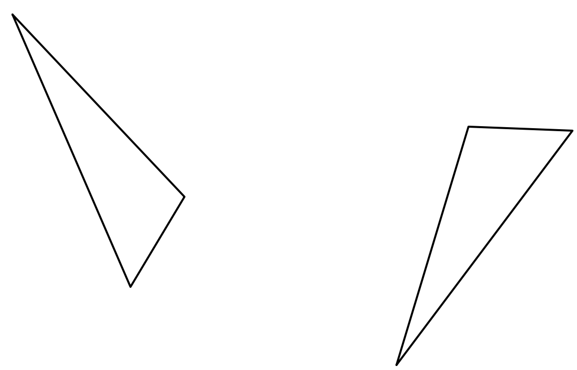
Problem 6.6.10 Suppose $z = z_1 z_2$ where $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$. Use Problem 6.6.9 to prove the angle sum formulas for sine and cosine.

7 Transformations, Congruence and Similarity

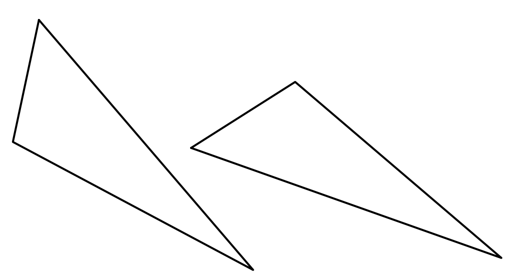
7.1 Congruence

Let's begin with some exercises to clarify the notion of *congruence*.

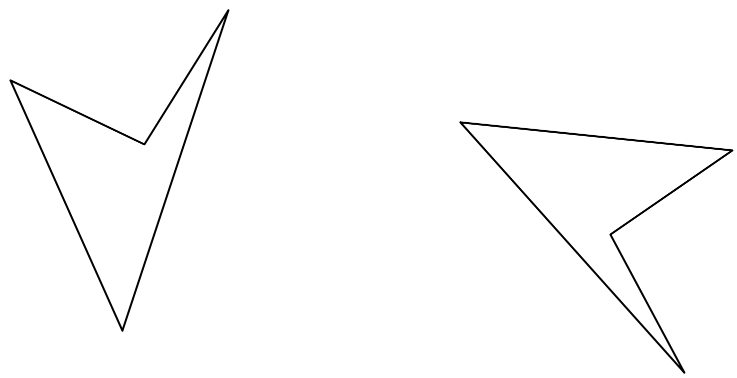
Problem 7.1.1 Are the following two figures congruent? Why or why not?



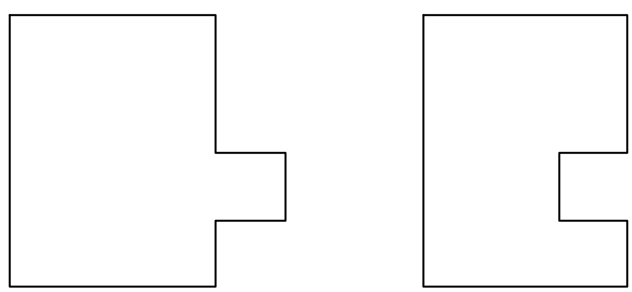
Problem 7.1.2 Are the following two figures congruent? Why or why not?



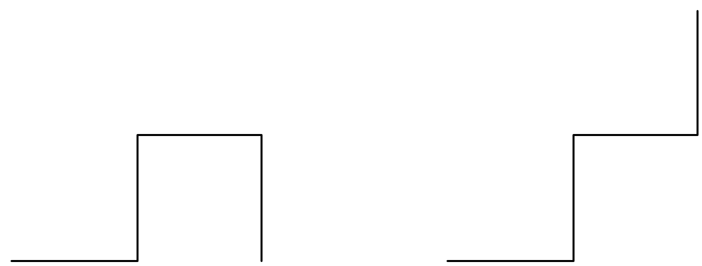
Problem 7.1.3 Are the following two figures congruent? Why or why not?



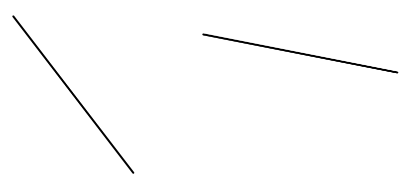
Problem 7.1.4 Are the following two figures congruent? Why or why not?



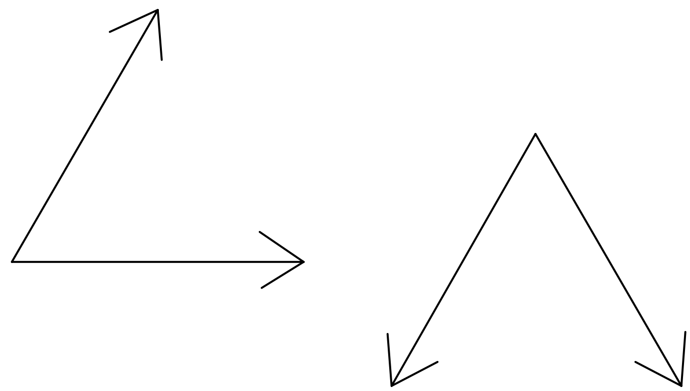
Problem 7.1.5 Are the following two figures congruent? Why or why not?



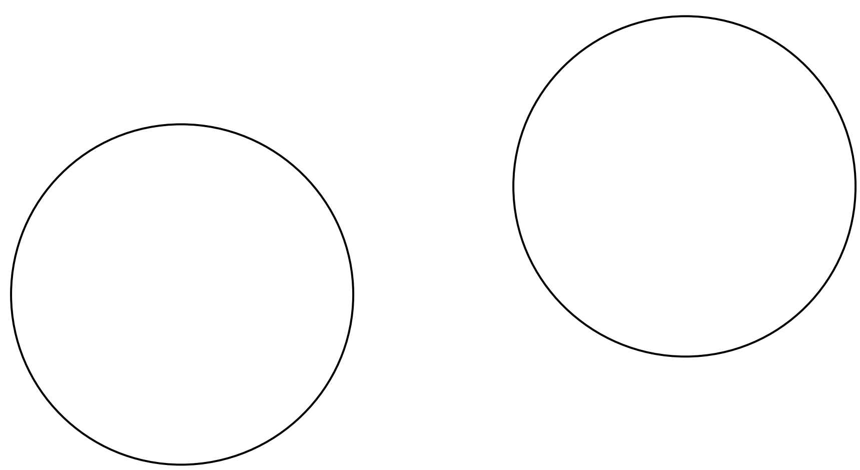
Problem 7.1.6 Are the following two figures congruent? Why or why not?



Problem 7.1.7 Are the following two figures congruent? (These are supposed to be rays.) Why or why not?



Problem 7.1.8 Are the following two figures congruent? Why or why not?



Problem 7.1.9 Let f be a function mapping the plane to itself. We call f a *bijection* if it

is one-to-one (i.e., an *injection*), and onto (i.e., a *surjection*). Suppose S_1 and S_2 are two subsets of the plane. Use the notion of bijections to define the notion of congruence between S_1 and S_2.

Problem 7.1.10 An *isometry* is a distance-preserving bijection f of the plane to itself. To say that f is *distance-preserving* means that the distance $f(P)f(Q)$ between the points $f(P)$ and $f(Q)$ equals the distance PQ between the points P and Q for all points P, Q in the plane. Let us *define* two subsets S_1, S_2 of the plane to be *congruent* if and only if there is an isometry f such that $f(S_1) = S_2$. Prove that if f is an isometry and A, B, C are any three points in the plane, then $m\angle f(A)f(B)f(C) = m\angle ABC$.

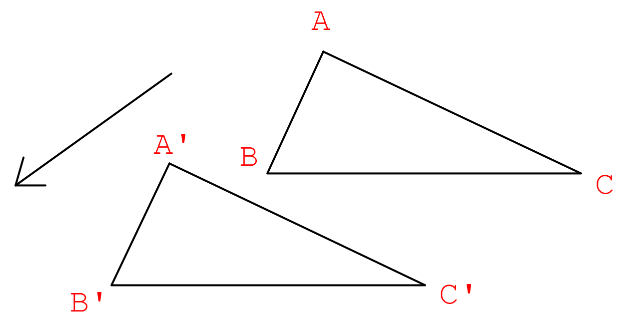
Problem 7.1.11 If two triangles are congruent under this new definition, explain why they are congruent under the "traditional" definition, of having congruent corresponding sides and congruent corresponding angles.

Problem 7.1.12 Explain why the new definition of congruence can be applied to answer all of the Exercises 7.1.1–7.1.8.

7.2 Isometries

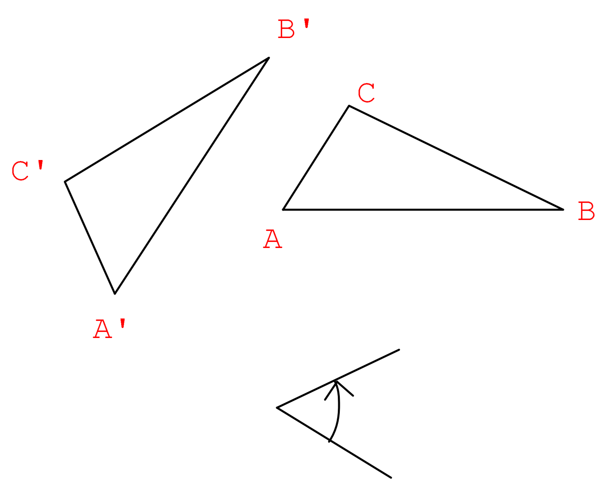
Recall the definition of isometry given in Problem 7.1.10. Four types of isometries are:

1. *Translation* by specified direction and amount. We can indicate the translation by drawing a vector.



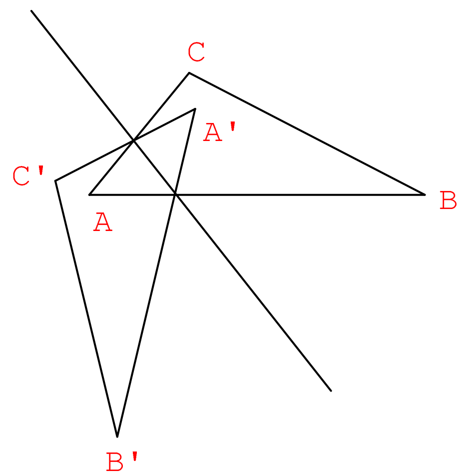
Note that the *identity isometry* is a special case of a translation in which the translation amount is zero.

2. *Rotation* by a specified angle about a specified point. We can indicate the rotation by drawing an angle at the center of rotation.

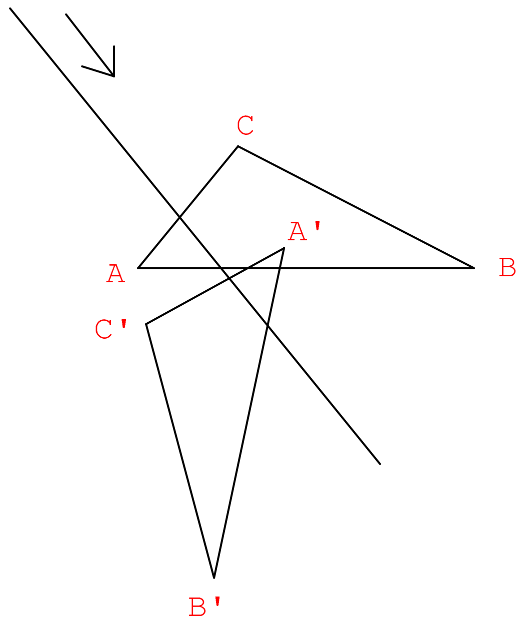


Note that the *identity isometry* is a special case of a rotation in which the rotation angle is zero.

3. *Reflection* across a specified line. We can indicate the reflection by drawing the line of reflection.

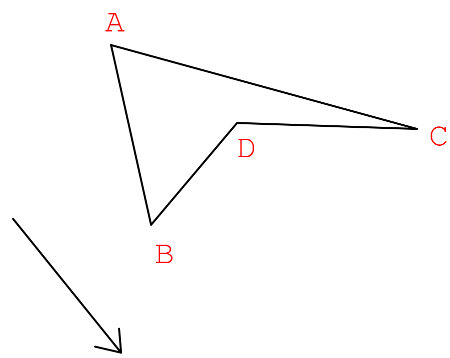


4. *Glide reflection*—reflection across a specified line followed by a translation parallel to that line by a specified amount. We can indicate the glide reflection by drawing the line and drawing a parallel vector. Remember that a reflection is a special case of a glide reflection in which the translation amount is zero.

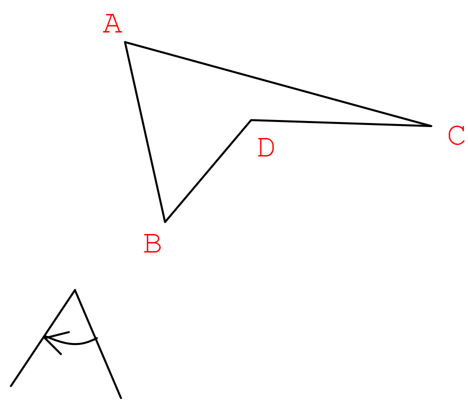


Problem 7.2.1 In each case below, apply the indicated isometry to the figure. You may need a protractor, compass, and straightedge. Note: You may also continue this exercise with a partner, with one person placing the figure and specifying the isometry, and the other applying the isometry to the figure.

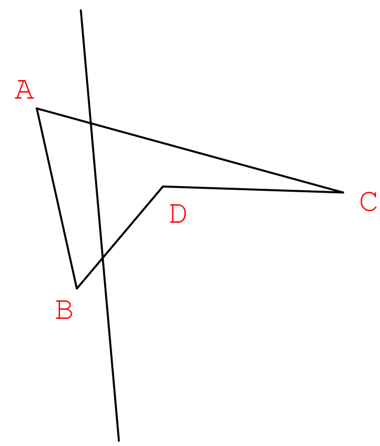
1.



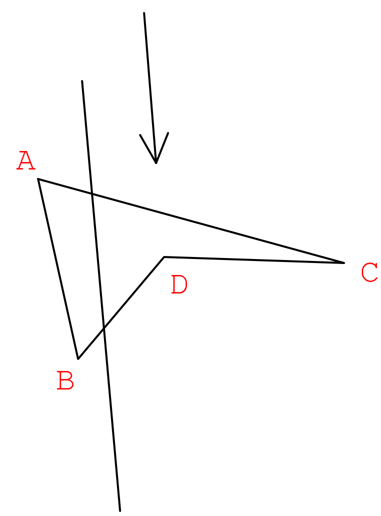
2.



3.



4.



Problem 7.2.2 Open each of the following GeoGebra files. In each case, drag the red figure around, observing the effect on the two figures. Determine what isometry maps the figure containing A to the second figure.

1. `Isometry 1`
2. `Isometry 2`
3. `Isometry 3`
4. `Isometry 4`

Problem 7.2.3 Open each of the following GeoGebra files. In each case, drag point A around, observing the effect on point A'. Determine what isometry maps point A to point A'.

1. `Isometry 5`
2. `Isometry 6`
3. `Isometry 7`
4. `Isometry 8`

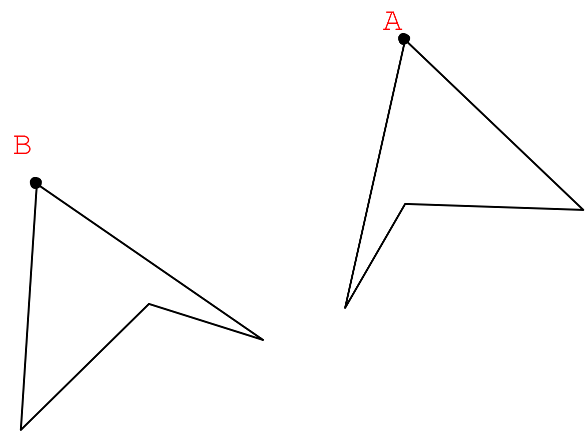
Problem 7.2.4 Mimic the above exercise by making "human isometries". Use the classroom floor as the plane, and specify an isometry f by drawing a translation vector, selecting a center and angle of rotation, choosing a reflection line, or choosing a reflection line and a parallel translation amount. Select two people, A and B, to play the role of the two points in the previous exercise. Person A moves around; person B must move to the correct position as specified by the isometry.

Problem 7.2.5 Learn how to use GeoGebra to apply isometries to various figures.

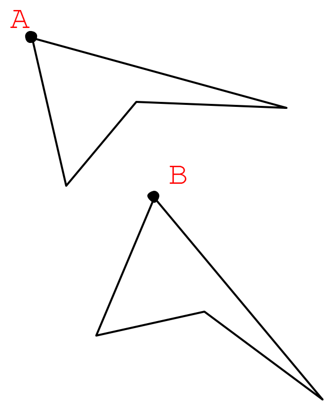
Problem 7.2.6 Go to the website for the National Library of Virtual Manipulatives, `http://nlvm.usu.edu`, and learn how to use "Transformations - Reflection", "Transformations - Rotation", and "Transformations - Translation". Write an explanation for the other members of your class.

Problem 7.2.7 In each of the following cases determine what isometry was applied to move the figure containing the point A to the figure containing the point B. If it is a translation, draw a vector of translation. If it is a rotation, mark the center of rotation and draw an angle of rotation. If it is a reflection, draw the line of reflection. If it is a glide reflection, draw the line of reflection and draw a vector of translation. Note: You may also extend this exercise with a partner, with one person placing the figures, either on paper or with Wingeom, and the other determining the isometry.

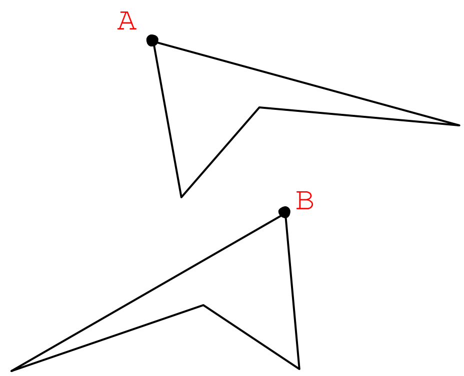
1.



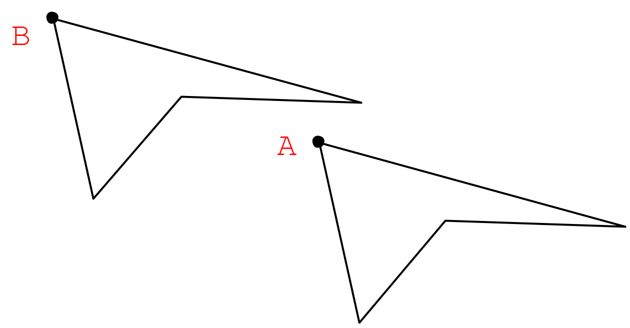
2.



3.



4.



Problem 7.2.8 Refer to Exercise 7.2.7. Describe general procedures to identify the isometry and its elements from such figures and justify your answers.

Problem 7.2.9 Cut out a pair of congruent scalene triangles. Place them on a piece of paper in any two locations. Convince yourself that in each case, one is related to the other by a translation, rotation, reflection, or glide reflection.

Problem 7.2.10 Given two different points A and B.

1. Describe all translations that map A to B.
2. Describe all rotations that map A to B.
3. Describe all reflections that map A to B.
4. Describe all glide reflections that map A to B.

Problem 7.2.11 Given three different points A, B, C such that $AB = AC$.

1. Describe all translations that map A to A and B to C.
2. Describe all rotations that map A to A and B to C.
3. Describe all reflections that map A to A and B to C.
4. Describe all glide reflections that map A to A and B to C.

Problem 7.2.12 Let f be an isometry of the plane (not necessarily one of the four specific types we have been discussing). Let A, B, C be three noncollinear points. Show that if you know $f(A)$, $f(B)$, and $f(C)$, then you can determine $f(P)$ for any point P. That is to say, f is uniquely determined by its action on any three particular noncollinear points.

Problem 7.2.13 Let f be an isometry of the plane (not necessarily one of the four specific types we have been discussing). Show that f can be expressed as a sequence (composition) of at most three reflections.

Problem 7.2.14 Show that any composition of at most three reflections is a translation, rotation, reflection, or glide reflection.

Problem 7.2.15 Here we examine the net result of performing two isometries in a row. Experiment on paper and with GeoGebra, and then justify your answers. What is the net result of:

1. A translation by (p_1, q_1) followed by a translation by (p_2, q_2)?
2. A reflection across a line ℓ_1 followed by a reflection across a parallel line ℓ_2?
3. A reflection across a line ℓ_1 followed by a reflection across a nonparallel line ℓ_2?
4. A rotation by α_1 about a point (p_1, q_1), followed by a rotation by α_2 about the same point?

5. A rotation by α_1 about a point (p_1, q_1), followed by a rotation by α_2 about a different point (p_2, q_2)?

Problem 7.2.16 Consider the following eight transformations:

- I, the identity transformation.
- R_{90}, rotation by 90 degrees counterclockwise about the origin.
- R_{180}, rotation by 180 degrees counterclockwise about the origin.
- R_{270}, rotation by 270 degrees counterclockwise about the origin.
- F_0, reflection ("flip") about the x-axis.
- F_{45}, reflection about the line $y = x$.
- F_{90}, reflection about the y-axis.
- F_{135}, reflection about the line $y = -x$.

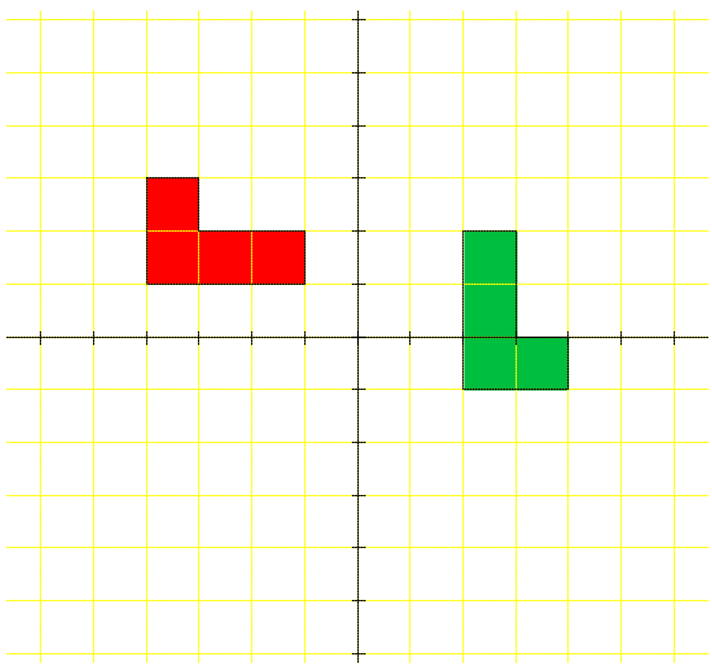
Fill in this multiplication table. The entry in row i, column j should be the net result of FIRST performing the transformation associated with COLUMN j, and THEN performing the transformation associated with ROW i.

$\circ$	I	R_{90}	R_{180}	R_{270}	F_0	F_{45}	F_{90}	F_{135}
I								
R_{90}								
R_{180}								
R_{270}								
F_0								
F_{45}								
F_{90}								
F_{135}								

7.3 Formulas for Isometries

Let's derive formulas for functions that transform a shape in certain ways.

Problem 7.3.1 Choose a partner. Construct two "L's" composed of four squares as indicated below in two different colors, say, green and red. One partner will place the two L's on grid paper in various ways, with vertices placed on grid points (they can even overlap!). The other will find a formula for a function, in the above form, that describes an isometry that will map the green L to the red L. Repeat this process several times, with partners changing roles.



Problem 7.3.2 Let's express the functions found in Exercise 7.3.1 in the form

$$\begin{aligned} x_2 &= a_1x_1 + b_1y_1 + c_1 \\ y_2 &= a_2x_1 + b_2y_1 + c_2 \end{aligned}$$

by listing the coefficients in a matrix like this:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & 0 & 1 \end{bmatrix}$$

This is handy because applying f to (x, y) corresponds to multiplying the matrix for f by the column vector

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

What common characteristics do you observe in the functions that you found?

Problem 7.3.3 Suppose f_1 and f_2 are two functions with corresponding matrices M_1 and M_2, respectively. Prove that the matrix for the composition function $f_2 \circ f_1$ (first performing f_1, then performing f_2) is the product of the matrices M_2M_1.

Problem 7.3.4 Determine the matrices for the following isometries:

1. Translation by the amount (p, q).
2. The *identity* isometry, $f(x, y) = (x, y)$, which may be regarded as a translation by the amount $(0, 0)$.
3. Rotation by 90 degrees about the point (p, q). Suggestion: First solve the case when $(p, q) = (0, 0)$. Then think of a rotation about an arbitrary point (p, q) as a composition of (1) a translation moving (p, q) to the origin, (2) a rotation about the origin, and (3) a translation that moves the origin back to the point (p, q).
4. Rotation by 180 degrees about the point (p, q).
5. Rotation by 270 degrees about the point (p, q).
6. Reflection across the horizontal line $y = p$.
7. Reflection across the vertical line $x = p$.

8. Reflection across the line $y = x + p$.
9. Reflection across the line $y = -x + p$.
10. Reflection across the horizontal line $y = p$ followed by a horizontal translation by q units.
11. Reflection across the vertical line $x = p$ followed by a vertical translation by q units.
12. Reflection across the line $y = x + p$ followed by a translation by (q, q) (parallel to the line). (A glide reflection.)
13. Reflection across the line $y = -x + p$ followed by a translation by $(q, -q)$ (parallel to the line). (A glide reflection.)

Problem 7.3.5 Choose a partner. One partner will write down the formula for a function that fits one of the above types, choosing specific numbers. The other partner will identify the type of isometry represented by the formula, describing the amount of translation, the center and amount of rotation, or the reflection line and the amount of translation parallel to that line. Repeat this procedure several times, with partners changing roles.

Problem 7.3.6 Take some of the formulas found in Exercise 7.3.1 and identify the isometry as in the previous exercise.

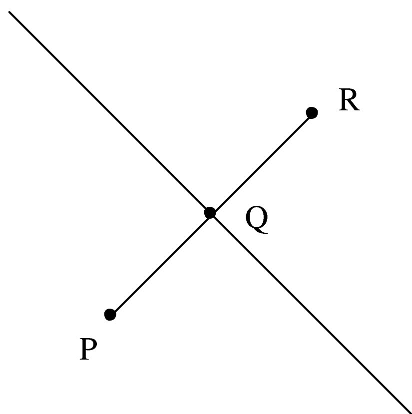
Problem 7.3.7 Recall the matrix for a rotation about the origin from Section 6.5. Now express it in the form of a 3×3 matrix. Find the matrices for the rotations about the origin by each angle that is a multiple of 30 degrees.

Problem 7.3.8 Find the matrices for the rotations about the origin by each angle that is a multiple of 45 degrees.

Problem 7.3.9 Find the matrix for the isometry that rotates by δ about the point (p, q). Suggestion: First apply a translation that moves the point (p, q) to the origin. Then rotate by δ about the origin. Finally apply a translation that moves the origin back to (p, q).

Problem 7.3.10 Confirm that the above matrix works for the rotation examples you considered in Exercise 7.3.4.

Problem 7.3.11 In this exercise we will derive the matrix for the reflection across the line with equation $px+qy+r=0$. Since this is the equation of a line, it cannot be the case that both p and q are zero. Divide the equation of the line through by $\sqrt{p^2+q^2}$ so that we may assume from now on that $p^2+q^2=1$.



To reflect the point $P=(x_1,y_1)$ across the line, we must move in a direction perpendicular to the line until we intersect the line at the point Q. Then we must move the same distance to the other side of the line, reaching the point R. The direction (p,q) is perpendicular to the line, so the coordinates of Q are $(x_1,y_1)+t(p,q)=(x_1+tp,y_1+tq)$ for some number t.

1. Substitute the point Q into the equation of the line and solve for t. Remember that we are assuming that $p^2+q^2=1$.
2. Find the coordinates of R, $(x_2,y_2)=(x_1,y_1)+2t(p,q)=(x_1+2tp,y_1+2tq)$.
3. Derive the matrix for this reflectional isometry.

4. Verify that the matrix is of the form

$$\begin{bmatrix} c & s & u \\ s & -c & v \\ 0 & 0 & 1 \end{bmatrix}$$

where $c^2 + s^2 = 1$. In what way is this different from the general rotation matrix?

Problem 7.3.12 Confirm that the above matrix works for the reflection examples you considered in Exercise 7.3.4.

Problem 7.3.13 Find the matrix for the isometry resulting from first reflecting across the line $px + qy + r = 0$ and then translating by $(tq, -tp)$ (which is a direction parallel to the line). This will be the matrix for the general glide reflection.

Problem 7.3.14 Confirm that the above matrix works for the glide reflection examples you considered in Exercise 7.3.4.

Problem 7.3.15 Prove that any matrix of the form

$$\begin{bmatrix} c & -s & u \\ s & c & v \\ 0 & 0 & 1 \end{bmatrix}$$

in which $c^2 + s^2 = 1$ is either a translation matrix or else corresponds to a rotation about some point. Suggestion: If the matrix is not a translation matrix, look at the results of Exercise 7.3.9 and prove that it is possible to solve for the angle of rotation, and for p and q.

Problem 7.3.16 Prove that any matrix of the form

$$\begin{bmatrix} c & s & u \\ s & -c & v \\ 0 & 0 & 1 \end{bmatrix}$$

in which $c^2 + s^2 = 1$, is either a reflection or a glide reflection matrix. Suggestion: Look at the results of Exercise 7.3.13 and prove that it is possible to solve for for p, q, r and t.

Problem 7.3.17 Find a partner. Take turns doing the following: One person chooses a particular isometry and derives the associated matrix. The other person reconstructs the isometry from the matrix.

Problem 7.3.18 Are there isometries of the form

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & 0 & 1 \end{bmatrix}$$

other than the ones we have found? Prove that if the above matrix corresponds to an isometry, then it must be of the form of a translation matrix, a rotation matrix, or a glide reflection matrix. Suggestion: Apply the isometry to the three points $A = (0,0)$, $B = (1,0)$, and $C = (0,1)$, remembering that an isometry must be distance-preserving.

Problem 7.3.19 Prove directly by algebra that each of the functions described by the matrices below are distance-preserving, assuming $c^2 + s^2 = 1$:

$$\begin{bmatrix} c & -s & u \\ s & c & v \\ 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} c & s & u \\ s & -c & v \\ 0 & 0 & 1 \end{bmatrix}$$

7.4 Transforming Functions

Refer to the sections "Transforming" and "Modeling" in the presentation here:

`http://www.ms.uky.edu/~lee/arkansas2011/Functions.pdf`

7.5 Basic Properties

Let's consider a set of properties of isometries and similarity transformations to be given, and see what we can prove from them. Some of these properties are strong assumptions

Note 7.5.1 Some properties of isometries:

1. An isometry maps points to points and lines to lines.
2. An isometry f preserves distance: If f maps A to A' and B to B', then $AB = A'B'$.
3. An isometry f preserves angle measure: If f maps A, B, C to A', B', C', respectively, then $\mathrm{m}\angle ABC = \mathrm{m}\angle A'B'C'$.

Note 7.5.2 Some properties of translations:

1. Every translation is an isometry.
2. Given any two different points A, B there is a (unique) translation that maps A to B.
3. If a translation maps a line ℓ to a line m, then either $\ell = m$ or ℓ is parallel to m.

Note 7.5.3 Some properties of reflections:

1. Every reflection is an isometry.
2. Every line is the axis of a (unique) reflection.
3. A point is fixed by a reflection if and only if it lies on the axis of that reflection.
4. Given any two different points A, B there is a (unique) reflection that maps A to B.

Note 7.5.4 Some properties of rotations:

1. Every rotation is an isometry.
2. Every point and angle measure determines a unique rotation about that point by that angle measure.
3. A point is fixed by a nonzero rotation if and only if it is the center of that rotation.
4. Given any three distinct points A, B, C such that $AC = BC$, there is a unique rotation about C that maps A to B.

Note 7.5.5 Some properties of similarity transformations:

1. A *similarity transformation* with *scaling factor* $s > 0$ is a bijection of the plane onto itself with the property that if A, B are mapped to A', B', respectively, then $A'B' = sAB$. That is to say, all distances are scaled by the same constant s.
2. A similarity transformation maps points to points and lines to lines.
3. An similarity transformation f preserves angle measure. If f maps A, B, C to A', B', C', respectively, then $\mathrm{m}\angle ABC = \mathrm{m}\angle A'B'C'$.

Note 7.5.6 Some properties of dilations:

1. Let P be a point and r be a nonzero real number. A *dilation* f centered at P with *dilation factor* r is described as follows: To find the image of a point A, if $r > 0$ choose point $B = f(A)$ on ray $\overrightarrow{PA}$ such that $PB = rPA$, and if $r < 0$ choose point $B = f(A)$ on the ray opposite ray $\overrightarrow{PA}$ such that $PB = |r|PA$.
2. Every dilation is a similarity transformation.
3. Every point P and nonzero real number r determines a unique dilation centered at that point.
4. If a dilation maps a line ℓ to a line m, then either $\ell = m$ or ℓ is parallel to m.

Theorem 7.5.7 *1. The composition of any two isometries is an isometry.*

2. *Every isometry is a composition of translations, reflections, and rotations.*
3. *Every isometry is a translation, reflection, rotation, or glide reflection.*
4. *Every translation is a composition of two reflections.*
5. *Every rotation is a composition of two reflections.*
6. *Every isometry is a composition of at most three reflections.*

Theorem 7.5.8 1. The composition of any two similarity transformations is a similarity transformation.

2. Every similarity transformation is a composition of a dilation and an isometry.

7.6 Some Consequences

Definition 7.6.1 Two figures are *congruent* if one is the image of the other under the action of an isometry. Two figures are *similar* if one is the image of the other under the action of a similarity transformation.

Problem 7.6.2 Two line segments are congruent if and only if they have the same measure. Two angles (each formed by two rays) are congruent if and only if they have the same measure.

Problem 7.6.3 Two triangles ΔABC and ΔDEF are congruent if and only if there is a one-to-one correspondence between the sets vertices, say, $A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$, such that corresponding sides have equal length and corresponding angles have equal measure.

Problem 7.6.4 Prove the SAS congruence criterion.

Problem 7.6.5 Prove the ASA congruence criterion.

Problem 7.6.6 Prove the SSS congruence criterion.

Problem 7.6.7 Prove that if $AB = AC$ in ΔABC, then $\text{m}\angle B = \text{m}\angle C$.

Problem 7.6.8 Prove that equilateral triangles are equiangular.

Problem 7.6.9 Prove that if $\text{m}\angle B = \text{m}\angle C$ in ΔABC, then $AB = AC$.

Problem 7.6.10 Prove that equiangular triangles are equilateral.

7.7 Symmetries of Frieze Patterns

Consider the set $S = \{(x, y) \in \mathbf{R}^2 : -1 \leq y \leq 1\}$. We will call this a *strip*. What are the possible symmetries of the strip?

1. The identity symmetry, which we will denote I.
2. Translations by any amount parallel to the x-axis. If we translate by an amount a (which may be positive, negative, or zero), we will denote this symmetry by T_a.
3. Rotations by 180° about a point on the x-axis. If we rotate about the point $(b, 0)$ we will denote this symmetry by R_b.
4. Vertical reflection across any line perpendicular to the x-axis. If we reflect across the line $x = c$, we will denote this symmetry by V_c.
5. Horizontal reflection across the x-axis, which we will denote H.
6. Glide reflection across the x-axis. If we reflect across the x-axis and then translate by an amount d (which may be positive, negative, or zero), we will denote this symmetry by G_d.

A *repeating strip pattern* or *frieze pattern* is a subset of the above strip that possesses one particular translational symmetry such that each of its translational symmetries is an integer repetition (positive, negative, or zero) of this particular one.

You can make some simple (boring) frieze patterns such as:

1. $\cdots$ `A A A A A` $\cdots$
2. $\cdots$ `O O O O O` $\cdots$
3. $\cdots$ `Z Z Z Z Z` $\cdots$

You can also find many examples on the web; e.g., search for "frieze pattern".

Problem 7.7.1 For each of the strip symmetries below, write a formula for the function. The first two are done for you.

1. $I(x, y) = (x, y)$.
2. $T_a(x, y) = (x + a, y)$.
3. $R_b(x, y) =$
4. $V_c(x, y) =$
5. $H(x, y) =$
6. $G_d(x, y) =$

Problem 7.7.2 For each of the following strip symmetries, write the formula. The first one is done for you.

1. A translation to the right by 2 units. $T_2(x, y) = (x + 2, y)$.
2. A translation to the left by 3 units.
3. A rotation by 180° about the point $(-5, 0)$.
4. A rotation by 180° about the point $(4, 0)$.
5. A reflection across the line $x = 10$.
6. A reflection across the line $x = -7$.
7. A glide reflection involving a translation to the right by 6 units.
8. A glide reflection involving a translation to the left by 8 units.

Problem 7.7.3 Identify the following strip symmetries. The first two are done for you.

1. $f(x, y) = (x, y)$. The identity: I.

2. $f(x, y) = (-x, y)$. Reflection about the vertical line $x = 0$: V_0.

3. $f(x, y) = (x, -y)$.

4. $f(x, y) = (-x, -y)$.

5. $f(x, y) = (x + 1, y)$.

6. $f(x, y) = (x - 2, y)$.

7. $f(x, y) = (-x + 1, y)$.

8. $f(x, y) = (-x - 2, y)$.

9. $f(x, y) = (x + 1, -y)$.

10. $f(x, y) = (x - 2, -y)$.

11. $f(x, y) = (-x + 1, -y)$.

12. $f(x, y) = (-x - 2, -y)$.

Problem 7.7.4 In the next set of problems, $f \circ g$ means FIRST perform g, THEN perform f. For the following, determine the formula, and then identify the symmetry:

1. $V_3 \circ H$.

2. $G_{-1} \circ V_2$.

3. $R_2 \circ G_1$.

4. $G_1 \circ R_2$.

5. $R_1 \circ R_3$.

6. $R_b \circ V_c$.

7. $T_a \circ R_b$.

8. $T_{-3} \circ R_0 \circ T_3$. Explain why this makes sense in a sentence or two.

9. $T_{-3} \circ V_0 \circ T_3$. Explain why this makes sense in a sentence or two.

10. Find f such that $f \circ T_2 = I$.
11. Find f such that $f \circ T_a = I$.
12. Find f such that $f \circ R_3 = I$.
13. Find f such that $f \circ R_b = I$.
14. Find f such that $f \circ V_{-1} = I$.
15. Find f such that $f \circ V_c = I$.
16. Find f such that $f \circ H = I$.
17. Find f such that $f \circ G_5 = I$.
18. Find f such that $f \circ G_d = I$.

Problem 7.7.5 Fill in the following composition table, filling in each entry with $f \circ g$ and identifying what kinds of symmetries can result. (You may have to redraw this chart to make it larger.) REMEMBER: $f \circ g$ means FIRST do g, and THEN do f.

f

g

$f \circ g$	I	T_b	R_b	V_b	H	G_b
I						
T_a						
R_a				G_{2b-2a} glide refl. (hor. refl. if $a = b$)		
V_a						
H						
G_a						

Problem 7.7.6 We will now try to classify repeating strip patterns by the types of symmetries they have. In the following table, R indicates the presence of some 180° rotational symmetry about a point on the x-axis, V indicates the presence of a reflectional symmetry across some vertical line, H indicates the presence of a reflectional symmetry across the x-axis, and G indicates the presence of some nontrivial glide reflectional symmetry across the x-axis (i.e., where the accompanying translation is by a non-zero amount, so it is not merely a horizontal reflection.) For a given row there are sixteen possible ways to place either a "Y" (for "Yes") or leave it blank (for "No") in each cell of the row (why?). Fill in the table with the sixteen possibilities. For each possibility, either draw a repeating strip pattern to the right of the row exhibiting precisely that combination of symmetries, or else use the results of the previous chart to briefly explain why that particular combination of symmetries is impossible for any repeating strip pattern. For example, the presence of H and T (all patterns have T) forces the presence of G, and the presence of R and V also forces

the presence of G.

T	R	V	H	G
Y				
Y				
Y				
Y				
Y				
Y				
Y				
Y				
Y				
Y				
Y				
Y				
Y				
Y				
Y				
Y				