USING MAPLE TO DETERMINE THE ROTATION MATRIX FOR A COUNTERCLOCKWISE ROTATION BY ANGLE t ABOUT THE AXIS [p,q,r] THROUGH THE ORIGIN

```
> with(LinearAlgebra):
```

Define the axis.

```
> axis1:=Matrix([[p],[q],[r]]);
```

$$axis1 := \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Write the matrix A1 that rotates about the z axis and zeroes out the y component of the axis.

```
> n1:=sqrt(p^2+q^2); A1:=Matrix([[p/n1,q/n1,0],[-q/n1,p/n1,0],[0,0,1]]);Determinant(A1);
```

$$n1 := \sqrt{p^2+q^2}$$

$$A1 := \begin{bmatrix} \frac{p}{\sqrt{p^2+q^2}} & \frac{q}{\sqrt{p^2+q^2}} & 0 \\ -\frac{q}{\sqrt{p^2+q^2}} & \frac{p}{\sqrt{p^2+q^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$1$$

```
> axis2:=simplify(Multiply(A1,axis1));
```

$$axis2 := \begin{bmatrix} \sqrt{p^2+q^2} \\ 0 \\ r \end{bmatrix}$$

Write the matrix A2 that rotates about the y axis and rotates the axis in the direction of the x axis.

```
> n2:=sqrt(p^2+q^2+r^2);
  A2:=Matrix([[axis2(1)/n2,0,axis2(3)/n2],[0,1,0],[-axis2(3)/n2,0,axis2(1)/n2]]);Determinant(
  A2);
```

$$n2 := \sqrt{p^2+q^2+r^2}$$

$$A2 := \begin{bmatrix} \frac{\sqrt{p^2+q^2}}{\sqrt{p^2+q^2+r^2}} & 0 & \frac{r}{\sqrt{p^2+q^2+r^2}} \\ 0 & 1 & 0 \\ -\frac{r}{\sqrt{p^2+q^2+r^2}} & 0 & \frac{\sqrt{p^2+q^2}}{\sqrt{p^2+q^2+r^2}} \end{bmatrix}$$

$$1$$

```
> axis3:=simplify(Multiply(A2,axis2));
```

$$axis3 := \begin{bmatrix} \sqrt{p^2+q^2+r^2} \\ 0 \\ 0 \end{bmatrix}$$

Write the matrix A3 that rotates by the angle t about the x axis.

```
> A3:=Matrix([[1,0,0],[0,cos(t),-sin(t)],[0,sin(t),cos(t)]]);
```

$$A3 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{bmatrix}$$

Calculate the matrix M = A1^(-1) A2^(-1) A3 A2 A1. This rotates the axis to the direction of the x axis, performs the rotation by the angle t, and then rotates the axis back to its original position.

```
> M:=simplify(Multiply(MatrixInverse(A1),Multiply(MatrixInverse(A2),Multiply(A3,Multiply(A2,A
  1)))));
```

$M :=$

$$\left[\frac{\cos(t)\,q^2+\cos(t)\,r^2+p^2}{p^2+q^2+r^2}, -\frac{\cos(t)\,p\,q+\sin(t)\sqrt{p^2+q^2+r^2}\,r-p\,q}{p^2+q^2+r^2},\right.$$

$$\left.-\frac{p\,r\sqrt{p^2+q^2+r^2}\cos(t)-\sin(t)\,p^2\,q-\sin(t)\,q^3-\sin(t)\,q\,r^2-p\,r\sqrt{p^2+q^2+r^2}}{(p^2+q^2+r^2)^{(3/2)}}\right]$$

$$\left[-\frac{\cos(t)\,p\,q-\sin(t)\sqrt{p^2+q^2+r^2}\,r-p\,q}{p^2+q^2+r^2}, \frac{\cos(t)\,p^2+\cos(t)\,r^2+q^2}{p^2+q^2+r^2},\right.$$

$$-\frac{q\,r\sqrt{p^2+q^2+r^2}\,\cos(t)+\sin(t)\,p^3+\sin(t)\,p\,q^2+\sin(t)\,p\,r^2-q\,r\sqrt{p^2+q^2+r^2}}{(p^2+q^2+r^2)^{(3/2)}}\Bigg]$$

$$\left[-\frac{\cos(t)\,r\,p+\sin(t)\,q\sqrt{p^2+q^2+r^2}-r\,p}{p^2+q^2+r^2},\ -\frac{\cos(t)\,r\,q-\sin(t)\,p\sqrt{p^2+q^2+r^2}-r\,q}{p^2+q^2+r^2},\ \frac{\cos(t)\,p^2+\cos(t)\,q^2+r^2}{p^2+q^2+r^2}\right]$$

Do some algebraic simplifications to get the equivalent matrix N.

```
> M(1,3);
```

$$-\frac{p\,r\sqrt{p^2+q^2+r^2}\,\cos(t)-\sin(t)\,p^2\,q-\sin(t)\,q^3-\sin(t)\,q\,r^2-p\,r\sqrt{p^2+q^2+r^2}}{(p^2+q^2+r^2)^{(3/2)}}$$

```
> N13:=-(p*r*cos(t)-sin(t)*q*n2-p*r)/n2^2;
```

$$N13 := -\frac{\cos(t)\,r\,p-\sin(t)\,q\sqrt{p^2+q^2+r^2}-r\,p}{p^2+q^2+r^2}$$

```
> M(1,3)-N13;
```

$$-\frac{p\,r\sqrt{p^2+q^2+r^2}\,\cos(t)-\sin(t)\,p^2\,q-\sin(t)\,q^3-\sin(t)\,q\,r^2-p\,r\sqrt{p^2+q^2+r^2}}{(p^2+q^2+r^2)^{(3/2)}}+\frac{\cos(t)\,r\,p-\sin(t)\,q\sqrt{p^2+q^2+r^2}-r\,p}{p^2+q^2+r^2}$$

```
> simplify(%);
```

$$0$$

```
> M(2,3);
```

$$-\frac{q\,r\sqrt{p^2+q^2+r^2}\,\cos(t)+\sin(t)\,p^3+\sin(t)\,p\,q^2+\sin(t)\,p\,r^2-q\,r\sqrt{p^2+q^2+r^2}}{(p^2+q^2+r^2)^{(3/2)}}$$

```
> N23:=-(q*r*cos(t)+sin(t)*p*n2-q*r)/(n2^2);
>
```

$$N23 := -\frac{\cos(t)\,r\,q+\sin(t)\,p\sqrt{p^2+q^2+r^2}-r\,q}{p^2+q^2+r^2}$$

```
> M(2,3)-N23;
```

$$-\frac{q\,r\sqrt{p^2+q^2+r^2}\,\cos(t)+\sin(t)\,p^3+\sin(t)\,p\,q^2+\sin(t)\,p\,r^2-q\,r\sqrt{p^2+q^2+r^2}}{(p^2+q^2+r^2)^{(3/2)}}+\frac{\cos(t)\,r\,q+\sin(t)\,p\sqrt{p^2+q^2+r^2}-r\,q}{p^2+q^2+r^2}$$

```
> simplify(%);
```

$$0$$

```
> N:=Matrix([[M[1,1],M[1,2],N13],[M[2,1],M[2,2],N23],[M[3,1],M[3,2],M[3,3]]]);
```

$$N := \begin{bmatrix} \frac{\cos(t)\,q^2+\cos(t)\,r^2+p^2}{p^2+q^2+r^2} & -\frac{\cos(t)\,p\,q+\sin(t)\sqrt{p^2+q^2+r^2}\,r-p\,q}{p^2+q^2+r^2} & -\frac{\cos(t)\,r\,p-\sin(t)\,q\sqrt{p^2+q^2+r^2}-r\,p}{p^2+q^2+r^2} \\ -\frac{\cos(t)\,p\,q-\sin(t)\sqrt{p^2+q^2+r^2}\,r-p\,q}{p^2+q^2+r^2} & \frac{\cos(t)\,p^2+\cos(t)\,r^2+q^2}{p^2+q^2+r^2} & -\frac{\cos(t)\,r\,q+\sin(t)\,p\sqrt{p^2+q^2+r^2}-r\,q}{p^2+q^2+r^2} \\ -\frac{\cos(t)\,r\,p+\sin(t)\,q\sqrt{p^2+q^2+r^2}-r\,p}{p^2+q^2+r^2} & -\frac{\cos(t)\,r\,q-\sin(t)\,p\sqrt{p^2+q^2+r^2}-r\,q}{p^2+q^2+r^2} & \frac{\cos(t)\,p^2+\cos(t)\,q^2+r^2}{p^2+q^2+r^2} \end{bmatrix}$$

Now assume that p^2+q^2+r^2=1 to make the formulas look nicer, getting matrix N2.

```
> N1:=N*(p^2+q^2+r^2);
```

$$N1 := \begin{bmatrix} \cos(t)\,q^2+\cos(t)\,r^2+p^2 & -\cos(t)\,p\,q-\sin(t)\sqrt{p^2+q^2+r^2}\,r+p\,q & -\cos(t)\,r\,p+\sin(t)\,q\sqrt{p^2+q^2+r^2}+r\,p \\ -\cos(t)\,p\,q+\sin(t)\sqrt{p^2+q^2+r^2}\,r+p\,q & \cos(t)\,p^2+\cos(t)\,r^2+q^2 & -\cos(t)\,r\,q-\sin(t)\,p\sqrt{p^2+q^2+r^2}+r\,q \\ -\cos(t)\,r\,p-\sin(t)\,q\sqrt{p^2+q^2+r^2}+r\,p & -\cos(t)\,r\,q+\sin(t)\,p\sqrt{p^2+q^2+r^2}+r\,q & \cos(t)\,p^2+\cos(t)\,q^2+r^2 \end{bmatrix}$$

```
> N2:=Matrix([[c*q^2+c*r^2+p^2,-c*p*q-s*r+p*q,-c*r*p+s*q+r*p],[-c*p*q+s*r+p*q,c*p^2+c*r^2+q^2
,-c*r*q-s*p+r*q],[-c*r*p-s*q+r*p,-c*r*q+s*p+r*q,c*p^2+c*q^2+r^2]]);
```

$$N2 := \begin{bmatrix} c\,q^2+c\,r^2+p^2 & -c\,p\,q+p\,q-r\,s & -c\,p\,r+p\,r+q\,s \\ -c\,p\,q+p\,q+r\,s & c\,p^2+c\,r^2+q^2 & -c\,q\,r-p\,s+q\,r \\ -c\,p\,r+p\,r-q\,s & -c\,q\,r+p\,s+q\,r & c\,p^2+c\,q^2+r^2 \end{bmatrix}$$

```
>
```

It turns out we can represent this formula using quaternions--expressions of the form a+bi+cj+dk, where i^2=j^2=k^2=ijk=-1.
First verify that if w=cos(t/2)+(pi+qj+rk)sin(t/2) where p^2+q^2+r^2=1, then w^(-1)=cos(t/2)=(pi+qj+rk)sin(t/2).
Then verify (takes longer) that if u=xi+yj+zk and u'=wuw^(-1), then the i,j,k coordinates of u' are the coordinates of u rotated by the angle t

about the axis [p.q.r] given by the matrix N2.